Variational resolution for some general classes of nonlinear evolutions. Part II

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Abstract

Using our results in [15], we provided existence theorems for the general classes of nonlinear evolutions. Finally, we give examples of applications of our results to parabolic, hyperbolic, Shrödinger, Navier-Stokes and other time-dependent systems of equations.

1 Introduction

Let X be a reflexive Banach space. Consider the following evolutional initial value problem:

$$\begin{cases} \frac{d}{dt}\big\{I\cdot u(t)\big\}+\Lambda_t\big(u(t)\big)=0 \qquad \text{in } (0,T_0),\\ I\cdot u(0)=v_0. \end{cases} \tag{1.1}$$

Here $I : X \to X^*$ (X^* is the space dual to X) is a fixed bounded linear inclusion operator, which we assume to be self-adjoint and strictly positive, $u(t) \in L^q\big((0,T_0);X\big)$ is an unknown function, such that $I\cdot u(t) \in W^{1,p}\big((0,T_0);X^*\big)$ (where $I\cdot h \in X^*$ is the value of the operator I at the point $h \in X$), $\Lambda_t(x) : X \to X^*$ is a fixed nonlinear mapping, considered for every fixed $t \in (0,T_0)$, and $v_0 \in X^*$ is a fixed initial value. The most trivial variational principle related to (1.1) is the following one. Consider some convex function $\Gamma(y) : X^* \to [0,+\infty)$, such that $\Gamma(y) = 0$ if and only if $y = 0$. Next define the following energy functional

$$E_0\big(u(\cdot)\big) := \int_0^{T_0}\Gamma\bigg(\frac{d}{dt}\big\{I\cdot u(t)\big\}+\Lambda_t\big(u(t)\big)\bigg)dt$$
$$\forall\, u(t)\in L^q\big((0,T_0);X\big)\ \text{ s.t. }\ I\cdot u(t)\in W^{1,p}\big((0,T_0);X^*\big)\ \text{ and }\ I\cdot u(0)=v_0\,. \tag{1.2}$$

Then it is obvious that $u(t)$ will be a solution to (1.1) if and only if $E_0\big(u(\cdot)\big) = 0$. Moreover, the solution to (1.1) will exist if and only if there exists a minimizer $u_0(t)$ of the energy $E_0(\cdot)$, which satisfies $E_0\big(u_0(\cdot)\big) = 0$.

We have the following generalization of this variational principle. Let $\Psi_t(x) : X \to [0,+\infty)$ be some convex Gateux differentiable function, considered for every fixed $t \in (0,T_0)$ and such that $\Psi_t(0) = 0$. Next define the Legendre transform of Ψ_t by

$$\Psi^*_t(y) := \sup\Big\{\big\langle z,y\big\rangle_{X\times X^*}-\Psi_t(z):\ z\in X\Big\}\qquad \forall y\in X^*\,. \tag{1.3}$$

It is well known that $\Psi^*_t(y) : X^* \to \mathbb{R}$ is a convex function and

$$\Psi_t(x)+\Psi^*_t(y)\ \geq\ \big\langle x,y\big\rangle_{X\times X^*}\qquad \forall\, x\in X,\, y\in X^*\,, \tag{1.4}$$

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with equality if and only if $y = D\Psi_t(x)$. Next for $\lambda \in \{0,1\}$ define the energy

$$E_\lambda\big(u\big) := \int\limits_0^{T_0} \left\{ \Psi_t\Big(\lambda u(t)\Big) + \Psi_t^*\bigg(-\frac{d}{dt}\big\{I\cdot u(t)\big\} - \Lambda_t\big(u(t)\big)\bigg) + \lambda\Big\langle u(t), \frac{d}{dt}\big\{I\cdot u(t)\big\} + \Lambda_t\big(u(t)\big)\Big\rangle_{X\times X^*}\right\} dt$$

$$\forall\, u(t) \in L^q\big((0,T_0);X\big) \text{ s.t. } I\cdot u(t) \in W^{1,p}\big((0,T_0);X^*\big) \text{ and } I\cdot u(0) = v_0. \quad (1.5)$$

Then, by (1.4) we have $E_\lambda(\,\cdot\,) \geq 0$ and moreover, $E_\lambda\big(u(\cdot)\big) = 0$ if and only if $u(t)$ is a solution to

$$\begin{cases} \frac{d}{dt}\big\{I\cdot u(t)\big\} + \Lambda_t\big(u(t)\big) + D\Psi_t\big(\lambda u(t)\big) = 0 & \text{in } (0,T_0), \\ I\cdot u(0) = v_0 \end{cases} \quad (1.6)$$

(note here that since $\Psi_t(0) = 0$, in the case $\lambda = 0$ (1.6) coincides with (1.1). Moreover, if $\lambda = 0$ then the energy defined in (1.2) is a particular case of the energy in (1.5), where we take $\Gamma(x) := \Psi^*(-x)$). So, as before, a solution to (1.6) exists if and only if there exists a minimizer $u_0(t)$ of the energy $E_\lambda(\cdot)$, which satisfies $E_\lambda\big(u_0(\cdot)\big) = 0$. Consequently, in order to establish the existence of solution to (1.6) we need to answer the following questions:

(a) Does a minimizer to the energy in (1.5) exist?

(b) Does the minimizer $u_0(t)$ of the corresponding energy $E_\lambda(\cdot)$ satisfies $E_\lambda\big(u_0(\cdot)\big) = 0$?

To the best of our knowledge, the energy in (1.5) with $\lambda = 1$, related to (1.6), was first considered for the heat equation and other types of evolutions by Brezis and Ekeland in [1]. In that work they also first asked question **(b)**: If we don't know a priori that a solution of the equation (1.6) exists, how to prove that the minimum of the corresponding energy is zero. This question was asked even for very simple PDE's like the heat equation. A detailed investigation of the energy of type (1.5), with $\lambda = 1$, was done in a series of works of N. Ghoussoub and his coauthors, see the book [7] and also [8], [9], [10], [11] . In these works they considered a similar variational principle, not only for evolutions but also for some other classes of equations. They proved some theoretical results about general self-dual variational principles, which in many cases, can provide the answer to questions **(a)+(b)** together and, consequently, to prove existence of solution for the related equations (see [7] for details). In [15] we provide an alternative approach to the questions **(a)** and **(b)**. We treat them separately and in particular, for question **(b)**, we derive the main information by studying the Euler-Lagrange equations for the corresponding energy. To our knowledge, such an approach was first considered in [14], and provided there an alternative proof of existence of solution for some initial value problems of parabolic systems. Generalizing this method, we provide in [15] with the answer to questions **(a)** and **(b)** for some wide classes of evolutions. In particular, regarding question **(b)**, we are able to prove that in some general cases not only the minimizer but also any critical point $u_0(t)$ (i.e. any solution of corresponding Euler-Lagrange equation) satisfies $E_\lambda\big(u_0(\cdot)\big) = 0$, i.e. is a solution to (1.6).

We can rewrite the definition of E_λ in (1.5) as follows. Since I is a self-adjoint and strictly positive operator, there exists a Hilbert space H and an injective bounded linear operator $T : X \to H$, whose image is dense in H, such that if we consider the linear operator $\widetilde{T} : H \to X^*$, defined by the formula

$$\big\langle x, \widetilde{T}\cdot y\big\rangle_{X\times X^*} := \big\langle T\cdot x, y\big\rangle_{H\times H} \qquad \text{for every } y \in H \text{ and } x \in X\,, \quad (1.7)$$

then we will have $\widetilde{T}\circ T \equiv I$, see Lemma 2.5 for details. We call $\{X, H, X^*\}$ an evolution triple with the corresponding inclusion operators $T : X \to H$ and $\widetilde{T} : H \to X^*$. Thus, if $v_0 = \widetilde{T}\cdot w_0$, for some $w_0 \in H$ and $p = q^* := q/(q-1)$, where $q > 1$, then we have

$$\int_0^{T_0} \Big\langle u(t), \frac{d}{dt}\big\{I\cdot u(t)\big\}\Big\rangle_{X\times X^*} dt = \frac{1}{2}\big\|T\cdot u(T_0)\big\|_H^2 - \frac{1}{2}\big\|w_0\big\|_H^2$$

(see Lemma 2.6 for details) and therefore,

$$E_\lambda(u) = J(u) :=$$
$$\int_0^{T_0}\left\{\Psi_t\Big(\lambda u(t)\Big)+\Psi_t^*\Big(-\frac{d}{dt}\{I\cdot u(t)\}-\Lambda_t\big(u(t)\big)\Big)+\lambda\Big\langle u(t),\Lambda_t\big(u(t)\big)\Big\rangle_{X\times X^*}\right\}dt+\frac{\lambda}{2}\big\|T\cdot u(T_0)\big\|_H^2-\frac{\lambda}{2}\big\|w_0\big\|_H^2$$
$$\forall\, u(t)\in L^q\big((0,T_0);X\big)\ \text{ s.t. }\ I\cdot u(t)\in W^{1,q^*}\big((0,T_0);X^*\big)\ \text{ and }\ I\cdot u(0)=\widetilde{T}\cdot w_0 \tag{1.8}$$

Our first main result in [15] provides the answer for question **(b)**, under some coercivity and growth conditions on Ψ_t and Λ_t:

Theorem 1.1. *Let $\{X,H,X^*\}$ be an evolution triple with the corresponding inclusion linear operators $T:X\to H$, which we assume to be injective and having dense image in H, $\widetilde{T}:H\to X^*$ be defined by (1.7) and $I:=\widetilde{T}\circ T:X\to X^*$. Next let $\lambda\in\{0,1\}$, $q\geq 2$, $p=q^*:=q/(q-1)$ and $w_0\in H$. Furthermore, for every $t\in[0,T_0]$ let $\Psi_t(x):X\to[0,+\infty)$ be a strictly convex function which is Gateaux differentiable at every $x\in X$, satisfying $\Psi_t(0)=0$ and the condition*

$$(1/C_0)\,\|x\|_X^q-C_0\leq\Psi_t(x)\leq C_0\,\|x\|_X^q+C_0\quad\forall x\in X,\ \forall t\in[0,T_0]\,, \tag{1.9}$$

for some $C_0>0$. We also assume that $\Psi_t(x)$ is a Borel function of its variables (x,t). Next, for every $t\in[0,T_0]$ let $\Lambda_t(x):X\to X^*$ be a function which is Gateaux differentiable at every $x\in X$, s.t. $\Lambda_t(0)\in L^{q^*}\big((0,T_0);X^*\big)$ and the derivative of Λ_t satisfies the growth condition

$$\|D\Lambda_t(x)\|_{\mathcal{L}(X;X^*)}\leq g\big(\|T\cdot x\|_H\big)\left(\|x\|_X^{q-2}+1\right)\quad\forall x\in X,\ \forall t\in[0,T_0]\,, \tag{1.10}$$

for some non-decreasing function $g(s):[0+\infty)\to(0,+\infty)$. We also assume that $\Lambda_t(x)$ is strongly Borel on the pair of variables (x,t) (see Definition 2.2). Assume also that Ψ_t and Λ_t satisfy the following monotonicity condition

$$\Big\langle h,\lambda\Big\{D\Psi_t\big(\lambda x+h\big)-D\Psi_t(\lambda x)\Big\}+D\Lambda_t(x)\cdot h\Big\rangle_{X\times X^*}\geq-\hat{g}\big(\|T\cdot x\|_H\big)\Big(\|x\|_X^q+\mu(t)\Big)\,\|T\cdot h\|_H^2$$
$$\forall x,h\in X,\ \forall t\in[0,T_0]\,, \tag{1.11}$$

for some non-decreasing function $\hat{g}(s):[0+\infty)\to(0,+\infty)$ and some nonnegative function $\mu(t)\in L^1\big((0,T_0);\mathbb{R}\big)$. Consider the set

$$\mathcal{R}_q:=\Big\{u(t)\in L^q\big((0,T_0);X\big):\ I\cdot u(t)\in W^{1,q^*}\big((0,T_0);X^*\big)\Big\}\,, \tag{1.12}$$

and the minimization problem

$$\inf\Big\{J(u):\ u(t)\in\mathcal{R}_q\ \ s.t\ \ I\cdot u(0)=\widetilde{T}\cdot w_0\Big\}\,, \tag{1.13}$$

where $J(u)$ is defined by (1.8). Then for every $u\in\mathcal{R}_q$ such that $I\cdot u(0)=\widetilde{T}\cdot w_0$ and for arbitrary function $h(t)\in\mathcal{R}_q$, such that $I\cdot h(0)=0$, the finite limit $\lim\limits_{s\to 0}\big(J(u+sh)-J(u)\big)/s$ exists. Moreover, for every such u the following four statements are equivalent:

(1) *u is a critical point of (1.13), i.e., for any function $h(t)\in\mathcal{R}_q$, such that $I\cdot h(0)=0$ we have*

$$\lim_{s\to 0}\frac{J(u+sh)-J(u)}{s}=0\,. \tag{1.14}$$

(2) *u is a minimizer to (1.13).*

(3) *$J(u)=0$.*

(4) *u is a solution to*

$$\begin{cases}\frac{d}{dt}\big\{I\cdot u(t)\big\}+\Lambda_t\big(u(t)\big)+D\Psi_t\big(\lambda u(t)\big)=0 \qquad \text{in } (0,T_0),\\ I\cdot u(0)=\widetilde{T}\cdot w_0.\end{cases} \tag{1.15}$$

Finally there exists at most one function $u\in\mathcal{R}_q$ which satisfies (1.15).

Remark 1.1. Assume that, instead of (1.11), one requires that Ψ_t and Λ_t satisfy the following inequality

$$\Big\langle h,\lambda\Big\{D\Psi_t\big(\lambda x+h\big)-D\Psi_t(\lambda x)\Big\}+D\Lambda_t(x)\cdot h\Big\rangle_{X\times X^*}\geq \\ \frac{\|h\|_X^2}{\tilde{g}(\|T\cdot x\|_H)}-\tilde{g}\big(\|T\cdot x\|_H\big)\cdot\Big(\|x\|_X^q+\mu(t)\Big)^{(2-r)/2}\cdot\|h\|_X^r\cdot\|T\cdot h\|_H^{(2-r)} \qquad \forall x,h\in X,\ \forall t\in[a,b]\,, \tag{1.16}$$

for some non-decreasing function $\tilde{g}(s):[0+\infty)\to(0,+\infty)$, some nonnegative function $\mu(t)\in L^1\big((0,T_0);\mathbb{R}\big)$ and some constant $r\in(0,2)$. Then (1.11) follows by the trivial inequality $(r/2)\,a^2+\big((2-r)/2\big)\,b^2\geq a^r\,b^{2-r}$.

Our first result in [15] about the existence of minimizer for $J(u)$ is the following Proposition:

Proposition 1.1. *Assume that $\{X,H,X^*\}$, $T,\widetilde{T},I$, λ,q,p, Ψ_t and Λ_t satisfy all the conditions of Theorem 1.1 together with the assumption $\lambda=1$. Moreover, assume that Ψ_t and Λ_t satisfy the following positivity condition*

$$\Psi_t(x)+\Big\langle x,\Lambda_t(x)\Big\rangle_{X\times X^*}\geq\frac{1}{\tilde{C}}\,\|x\|_X^q-\tilde{C}\Big(\|x\|_X^r+1\Big)\Big(\|T\cdot x\|_H^{(2-r)}+1\Big)-\bar{\mu}(t) \quad \forall x\in X,\ \forall t\in[0,T_0], \tag{1.17}$$

where $r\in[0,2)$ and $\tilde{C}>0$ are some constants and $\bar{\mu}(t)\in L^1\big((0,T_0);\mathbb{R}\big)$ is some nonnegative function. Furthermore, assume that

$$\Lambda_t(x)=A_t\big(S\cdot x\big)+\Theta_t(x) \qquad \forall\, x\in X,\ \forall\, t\in[0,T_0], \tag{1.18}$$

where Z is a Banach space, $S:X\to Z$ is a compact operator and for every $t\in[0,T_0]$ $A_t(z):Z\to X^*$ is a function which is strongly Borel on the pair of variables (z,t) and Gateaux differentiable at every $z\in Z$, $\Theta_t(x):X\to X^*$ is strongly Borel on the pair of variables (x,t) and Gateaux differentiable at every $x\in X$, $\Theta_t(0),A_t(0)\in L^{q^*}\big((0,T_0);X^*\big)$ and the derivatives of A_t and Θ_t satisfy the growth condition

$$\|D\Theta_t(x)\|_{\mathcal{L}(X;X^*)}+\|DA_t(S\cdot x)\|_{\mathcal{L}(Z;X^*)}\leq g\big(\|T\cdot x\|\big)\left(\|x\|_X^{q-2}+1\right) \quad \forall x\in X,\ \forall t\in[0,T_0] \tag{1.19}$$

for some nondecreasing function $g(s):[0,+\infty)\to(0+\infty)$. Next assume that for every sequence $\big\{x_n(t)\big\}_{n=1}^{+\infty}\subset L^q\big((0,T_0);X\big)$ such that the sequence $\big\{I\cdot x_n(t)\big\}$ is bounded in $W^{1,q^*}\big((0,T_0);X^*\big)$ and $x_n(t)\rightharpoonup x(t)$ weakly in $L^q\big((0,T_0);X\big)$ we have

- $\Theta_t\big(x_n(t)\big)\rightharpoonup\Theta_t\big(x(t)\big)$ *weakly in* $L^{q^*}\big((0,T_0);X^*\big)$,
- $\underline{\lim}_{n\to+\infty}\int_0^{T_0}\Big\langle x_n(t),\Theta_t\big(x_n(t)\big)\Big\rangle_{X\times X^*}dt\geq\int_0^{T_0}\Big\langle x(t),\Theta_t\big(x(t)\big)\Big\rangle_{X\times X^*}dt.$

Finally let $w_0\in H$ be such that $w_0=T\cdot u_0$ for some $u_0\in X$, or more generally, $w_0\in H$ be such that $\mathcal{A}_{w_0}:=\big\{u\in\mathcal{R}_q:\ I\cdot u(0)=\widetilde{T}\cdot w_0\big\}\neq\emptyset$. Then there exists a minimizer to (1.13).

As a consequence of Theorem 1.1 and Proposition 1.1 we have the following Corollary.

Corollary 1.1. *Assume that we are in the settings of Proposition 1.1. Then there exists a unique solution $u(t) \in \mathcal{R}_q$ to*

$$\begin{cases}\frac{d}{dt}\big\{I\cdot u(t)\big\}+\Lambda_t\big(u(t)\big)+D\Psi_t\big(u(t)\big)=0 \qquad \text{in } (0,T_0),\\ I\cdot u(0)=\widetilde{T}\cdot w_0.\end{cases} \tag{1.20}$$

In this paper using Corollary 1.1 as a basis, by the appropriate approximation, we obtain further existence Theorems, under much weaker assumption on coercivity and compactness. The first Theorem improves the existence part of Corollary 1.1. (see Theorem 3.2 as an equivalent formulation and Theorem 3.3 as an important particular case).

Theorem 1.2. *Let $q\geq 2$ and $\{X,H,X^*\}$ be an evolution triple with the corresponding inclusion linear operators $T:X\to H$, which we assume to be injective and having dense image in H, $\widetilde{T}:H\to X^*$, defined by (1.7), and $I:=\widetilde{T}\circ T:X\to X^*$. Assume also that the Banach space X is separable. Furthermore, for every $t\in[0,T_0]$ let $\Psi_t(x):X\to[0,+\infty)$ be a convex function which is Gateaux differentiable at every $x\in X$, satisfies $\Psi_t(0)=0$ and satisfies the growth condition*

$$0\leq \Psi_t(x)\leq C\,\|x\|_X^q+C \quad \forall x\in X,\ \forall t\in[0,T_0]\,, \tag{1.21}$$

for some $C>0$. We also assume that $\Psi_t(x)$ is Borel on the pair of variables (x,t). Furthermore, for every $t\in[0,T_0]$ let $\Lambda_t(x):X\to X^*$ be a function which is Gateaux differentiable at every $x\in X$, $\Lambda_t(0)\in L^{q^*}\big((0,T_0);X^*\big)$ and the derivative of Λ_t satisfies the growth condition

$$\|D\Lambda_t(x)\|_{\mathcal{L}(X;X^*)}\leq g\big(\|T\cdot x\|_H\big)\left(\|x\|_X^{q-2}+1\right) \quad \forall x\in X,\ \forall t\in[0,T_0]\,, \tag{1.22}$$

for some nondecreasing function $g(s):[0,+\infty)\to(0,+\infty)$. We also assume that $\Lambda_t(x)$ is Borel on the pair of variables (x,t). Assume also that Λ_t and Ψ_t satisfy the following monotonicity and positivity conditions

$$\Big\langle h,\Big\{D\Psi_t(x+h)-D\Psi_t(x)\Big\}+D\Lambda_t(x)\cdot h\Big\rangle_{X\times X^*}\geq \\ -\hat{g}\big(\|T\cdot x\|_H\big)\cdot\Big(\|x\|_X^q+\mu(t)\Big)^{(2-r)/2}\cdot\|h\|_X^r\cdot\|T\cdot h\|_H^{(2-r)} \qquad \forall x\in X,\ \forall h\in X\ \forall t\in[0,T_0]\,, \tag{1.23}$$

and

$$\Big\langle x,D\Psi_t(x)+\Lambda_t(x)\Big\rangle_{X\times X^*}\geq\frac{1}{\hat{C}}\,\|x\|_X^q-\hat{C}\Big(\|x\|_X^r+1\Big)\Big(\|T\cdot x\|_H^{(2-r)}+1\Big)-\mu(t) \qquad \forall x\in X,\ \forall t\in[0,T_0]\,, \tag{1.24}$$

where $r\in[0,2)$, $\hat{g}(s):[0,+\infty)\to(0,+\infty)$ is some nondecreasing function, $\mu(t)\in L^1\big((0,T_0);\mathbb{R}\big)$ is some nonnegative function and $\hat{C}>0$ is some constant. Finally assume that the mapping $\Gamma\big(x(t)\big):\big\{\zeta(t)\in L^q(a,b;X):\ T\cdot\zeta(t)\in L^\infty(a,b;H)\big\}\to L^{q^*}\big((0,T_0);X^*\big)$, defined by

$$\Big\langle h(t),\Gamma\big(x(t)\big)\Big\rangle_{L^q((0,T_0);X)\times L^{q^*}((0,T_0);X^*)}:=\int_0^{T_0}\Big\langle h(t),\Lambda_t\big(x(t)\big)\Big\rangle_{X\times X^*}dt \\ \forall\, x(t)\in\big\{\zeta(t)\in L^q(a,b;X):\ T\cdot\zeta(t)\in L^\infty(a,b;H)\big\},\ \forall\, h(t)\in L^q\big((0,T_0);X\big)\,, \tag{1.25}$$

satisfies the following compactness property. For every sequence $\big\{u_n(t)\big\}_{n=1}^{+\infty}\subset L^q\big((0,T_0);X\big)$, such that $\big\{T\cdot u_n(t)\big\}_{n=1}^{+\infty}\subset L^\infty\big((0,T_0);H\big)$, $u_n(t)\rightharpoonup u(t)$ weakly in $L^q\big((0,T_0);X\big)$, $\big\{T\cdot u_n(t)\big\}_{n=1}^{+\infty}$ is bounded in $L^\infty\big((0,T_0);H\big)$ and $T\cdot u_n(t)\rightharpoonup T\cdot u(t)$ weakly in H for a.e. $t\in(0,T_0)$, the following conditions are satisfied:

- $\underline{\lim}_{n\to+\infty}\Big\langle u_n-u,\Gamma(u_n)\Big\rangle_{L^q((0,T_0);X)\times L^{q^*}((0,T_0);X^*)}\geq 0.$

- If we have $\lim_{n\to+\infty}\Big\langle u_n-u,\Gamma(u_n)\Big\rangle_{L^q((0,T_0);X)\times L^{q^*}((0,T_0);X^*)}=0$, then necessarily $\Gamma(u_n)\rightharpoonup\Gamma(u)$ weakly in $L^{q^*}\big((0,T_0);X^*\big)$.

Then for every $w_0\in H$ there exists $u(t)\in L^q\big((0,T_0);X\big)$, such that $I\cdot\big(u(t)\big)\in W^{1,q^*}\big((0,T_0);X^*\big)$, where $q^*:=q/(q-1)$, and $u(t)$ is a solution to (1.20).

The second existence result is useful in the study of Parabolic, Hyperbolic, Parabolic-Hyperbolic, Shrödinger, Navier-Stokes and other types of equations (see Theorem 3.4 as an equivalent formulation, and Theorem 3.5 and Corollary 3.1, as important particular cases).

Theorem 1.3. *Let $q\geq 2$ and let X and Z be reflexive Banach spaces and X^* and Z^* be the corresponding dual spaces. Furthermore let H be a Hilbert space. Suppose that $Q:X\to Z$ is an injective bounded linear operator such that its image is dense on Z. Furthermore, suppose that $P:Z\to H$ is an injective bounded linear operator such that its image is dense on H. Let $T:X\to H$ be defined by $T:=P\circ Q$. So that $\{X,H,X^*\}$ is an evolution triple with the corresponding inclusion operators $T:X\to H$, $\widetilde{T}:H\to X^*$ defined by (1.7) and $I:=\widetilde{T}\circ T$. Assume also that the Banach space X is separable. Furthermore, for every $t\in[0,T_0]$ let $\Lambda_t(z):Z\to X^*$ and $A_t(z):Z\to X^*$ be functions which are Gateaux differentiable at every $z\in Z$ and $A_t(0),\Lambda_t(0)\in L^{q^*}\big((0,T_0);X^*\big)$. Assume that for every $t\in[0,T]$ they satisfy the following bounds*

$$\big\|D\Lambda_t(z)\big\|_{\mathcal{L}(Z;X^*)}\leq g\big(\|P\cdot z\|_H\big)\cdot\Big(\|z\|_Z^{q-2}+1\Big)\qquad\forall z\in Z,\ \forall t\in[0,T_0]\,,\tag{1.26}$$

$$\big\|\Lambda_t(z)\big\|_{X^*}\leq g\big(\|P\cdot z\|_H\big)\cdot\Big(\|L_0\cdot z\|_{V_0}^{q-1}+1\Big)\qquad\forall z\in Z,\ \forall t\in[0,T_0]\,,\tag{1.27}$$

and

$$\big\|DA_t(z)\big\|_{\mathcal{L}(Z;X^*)}\leq g\big(\|P\cdot z\|_H\big)\cdot\Big(\|L_0\cdot z\|_{V_0}^{q-2}+1\Big)\qquad\forall z\in Z,\ \forall t\in[0,T_0]\,,\tag{1.28}$$

where $g(s):[0,+\infty)\to(0,+\infty)$ is some nondecreasing function, V_0 is some Banach space and $L_0:Z\to V_0$ is some compact linear operator. Moreover, assume that Λ_t and A_t satisfy the following monotonicity and positivity conditions

$$\Big\langle h,DA_t(z)\cdot(Q\cdot h)+D\Lambda_t(z)\cdot(Q\cdot h)\Big\rangle_{X\times X^*}\geq-\hat{g}\big(\|P\cdot z\|_H\big)\cdot\Big(\|z\|_Z^q+\mu(t)\Big)^{(2-r)/2}\cdot\big\|h\big\|_X^r\cdot\big\|T\cdot h\big\|_H^{(2-r)}\qquad\forall z\in Z,\ \forall h\in X,\ \forall t\in[0,T_0],\tag{1.29}$$

and

$$\Big\langle h,A_t(Q\cdot h)+\Lambda_t(Q\cdot h)\Big\rangle_{X\times X^*}\geq\big(1/\bar{C}\big)\big\|Q\cdot h\big\|_Z^q-\bar{C}\Big(\big\|Q\cdot h\big\|_Z^r+1\Big)\cdot\Big(\big\|T\cdot h\big\|_H^{(2-r)}+1\Big)-\mu(t)\qquad\forall h\in X\,,\ \forall t\in[0,T_0]\,,\tag{1.30}$$

where $r\in[0,2)$, $\hat{g}(s):[0,+\infty)\to(0,+\infty)$ is some nondecreasing function, $\mu(t)\in L^1\big((0,T_0);\mathbb{R}\big)$ is some nonnegative function and $\bar{C}>0$ is some constant. We also assume that $\Lambda_t(z)$ $A_t(z)$ are Borel on the pair of variables (z,t). Finally assume that there exists a family of Banach spaces $\{V_j\}_{j=1}^{+\infty}$ and a family of compact bounded linear operators $\{L_j\}_{j=1}^{+\infty}$, where $L_j:Z\to V_j$, which satisfy the following condition:

- *If $\{h_n\}_{n=1}^{+\infty}\subset Z$ is a sequence and $h_0\in Z$, are such that for every fixed j $\lim_{n\to+\infty}L_j\cdot h_n=L_j\cdot h_0$ strongly in V_j and $P\cdot h_n\rightharpoonup P\cdot h_0$ weakly in H, then for every fixed $t\in(0,T_0)$ we have $\Lambda_t(h_n)\rightharpoonup\Lambda_t(h_0)$ weakly in X^* and $DA_t(h_n)\to DA_t(h_0)$ strongly in $\mathcal{L}(Z,X^*)$.*

Then for every $w_0\in H$ there exists $z(t)\in L^q\big((0,T_0);Z\big)$ such that $w(t):=P\cdot z(t)\in L^\infty\big((0,T_0);H\big)$, $v(t):=\widetilde{T}\cdot\big(w(t)\big)\in W^{1,q^*}\big((0,T_0);X^*\big)$ and $z(t)$ satisfies the following equation

$$\begin{cases}\frac{dv}{dt}(t)+A_t\big(z(t)\big)+\Lambda_t\big(z(t)\big)=0 & \text{for a.e. } t\in(0,T_0),\\ v(a)=\widetilde{T}\cdot w_0.\end{cases}\tag{1.31}$$

On section 4 we give examples of the applications of Theorems 1.2 and 1.3, providing the existence results for various classes of time dependent partial differential equations including parabolic, hyperbolic, Shrödinger and Navier-Stokes systems.

2 Notations and preliminaries

Throughout the paper by the linear space we mean the real linear space.

- For given normed space X we denote by X^* the dual space (the space of continuous (bounded) linear functionals from X to $\mathbb{R}$).
- For given $h \in X$ and $x^* \in X^*$ we denote by $\left\langle h, x^* \right\rangle_{X \times X^*}$ the value in $\mathbb{R}$ of the functional x^* on the vector h.
- For given two normed linear spaces X and Y we denote by $\mathcal{L}(X;Y)$ the linear space of continuous (bounded) linear operators from X to Y.
- For given $A \in \mathcal{L}(X;Y)$ and $h \in X$ we denote by $A \cdot h$ the value in Y of the operator A on the vector h.
- We set $\|A\|_{\mathcal{L}(X;Y)} = \sup\{\|A \cdot h\|_Y : \ h \in X, \ \|h\|_X \leq 1\}$. Then it is well known that $\mathcal{L}(X;Y)$ will be a normed linear space. Moreover $\mathcal{L}(X;Y)$ will be a Banach space if Y is a Banach space.

Definition 2.1. Let X and Y be two normed linear spaces. We say that a function $F : X \to Y$ is Gateaux differentiable at the point $x \in X$ if there exists $A \in \mathcal{L}(X;Y)$ such that the following limit exists in Y and satisfy,

$$\lim_{s \to 0} \frac{1}{s}\Big(F(x + sh) - F(x)\Big) = A \cdot h \quad \forall h \in X\,.$$

In this case we denote the operator A by $DF(x)$ and the value $A \cdot h$ by $DF(x) \cdot h$.

Next we remind some Definitions and Lemmas of [15]. Part of them are well known. The proves of all the following Lemmas can be found in [15].

Definition 2.2. Let X and Y be two normed linear spaces and $U \subset X$ be a Borel subset. We say that the mapping $F(x) : U \to Y$ is strongly Borel if the following two conditions are satisfied.

- F is a Borel mapping i.e. for every Borel set $W \subset Y$, the set $\{x \in U : \ F(x) \in W\}$ is also Borel.
- For every separable subspace $X' \subset X$, the set $\{y \in Y \ : \ y = F(x), \ x \in U \cap X'\}$ is also contained in some separable subspace of Y.

Definition 2.3. For a given Banach space X with the associated norm $\|\cdot\|_X$ and a real interval (a, b) we denote by $L^q(a, b; X)$ the linear space of (equivalence classes of) strongly measurable (i.e equivalent to some strongly Borel mapping) functions $f : (a, b) \to X$ such that the functional

$$\|f\|_{L^q(a,b;X)} := \begin{cases} \Big(\int_a^b \|f(t)\|_X^q dt\Big)^{1/q} & \text{if } \ 1 \leq q < \infty \\ \operatorname{es\,sup}_{t \in (a,b)} \|f(t)\|_X & \text{if } \ q = \infty \end{cases}$$

is finite. It is known that this functional defines a norm with respect to which $L^q(a, b; X)$ becomes a Banach space. Moreover, if X is reflexive and $1 < q < \infty$ then $L^q(a, b; X)$ will be a reflexive space with the corresponding dual space $L^{q^*}(a, b; X^*)$, where $q^* = q/(q-1)$. It is also well known that the subspace of continuous functions $C^0([a, b]; X) \subset L^q(a, b; X)$ is dense i.e. for every $f(t) \in L^q(a, b; X)$ there exists a sequence $\{f_n(t)\} \subset C^0([a, b]; X)$ such that $f_n(t) \to f(t)$ in the strong topology of $L^q(a, b; X)$.

We will need the following simple Lemma.

Definition 2.4. Let X be a reflexive Banach space and let (a,b) be a finite real interval. We say that $v(t) \in L^q(a,b;X)$ belongs to $W^{1,q}(a,b;X)$ if there exists $f(t) \in L^q(a,b;X)$ such that for every $\delta(t) \in C^1\big((a,b);X^*\big)$ satisfying $\operatorname{supp}\delta \subset\subset (a,b)$ we have

$$\int_a^b \big\langle f(t), \delta(t)\big\rangle_{X\times X^*} dt = -\int_a^b \Big\langle v(t), \frac{d\delta}{dt}(t)\Big\rangle_{X\times X^*} dt\,.$$

In this case we denote $f(t)$ by $v'(t)$ or by $\frac{dv}{dt}(t)$. It is well known that if $v(t) \in W^{1,1}(a,b;X)$ then $v(t)$ is a bounded and continuous function on $[a,b]$ (up to a redefining of $v(t)$ on a subset of $[a,b]$ of Lebesgue measure zero), i.e. $v(t) \in C^0\big([a,b];X\big)$ and for every $\delta(t) \in C^1\big([a,b];X^*\big)$ and every subinterval $[\alpha,\beta] \subset [a,b]$ we have

$$\int_\alpha^\beta \bigg\{\Big\langle \frac{dv}{dt}(t), \delta(t)\Big\rangle_{X\times X^*} + \Big\langle v(t), \frac{d\delta}{dt}(t)\Big\rangle_{X\times X^*}\bigg\}dt = \big\langle v(\beta), \delta(\beta)\big\rangle_{X\times X^*} - \big\langle v(\alpha), \delta(\alpha)\big\rangle_{X\times X^*}\,. \tag{2.1}$$

Lemma 2.1. *Let X and Y be two reflexive Banach spaces, $S \in \mathcal{L}(X,Y)$ be an injective inclusion (i.e. it satisfies $\ker S = 0$) and (a,b) be a finite real interval. Then if $u(t) \in L^q(a,b;X)$ is such that $v(t) := S\cdot u(t) \in W^{1,q}(a,b;Y)$ and there exists $f(t) \in L^q(a,b;X)$ such that $\frac{dv}{dt}(t) = S\cdot f(t)$ then $u(t) \in W^{1,q}(a,b;X)$ and $\frac{du}{dt}(t) = f(t)$.*

Definition 2.5. Let X be a Banach space. We say that a function $\Psi(x) : X \to \mathbb{R}$ is convex (strictly convex) if for every $\lambda \in (0,1)$ and for every $x,y \in X$ s.t. $x \neq y$ we have

$$\Psi\Big(\lambda x + (1-\lambda)y\Big) \;\;\leq\;\; (\;<\;) \;\;\; \lambda\Psi(x) + (1-\lambda)\Psi(y)\,.$$

It is well known that if $\Psi(x) : X \to \mathbb{R}$ is a convex (strictly convex) function which is Gateaux differentiable at every $x \in X$ then for every $x,y \in X$ s.t. $x \neq y$ we have

$$\Psi(y) \;\;\geq\;\; (\;>\;) \;\;\; \Psi(x) + \Big\langle y - x, D\Psi(x)\Big\rangle_{X\times X^*}\,, \tag{2.2}$$

and

$$\Big\langle y - x, D\Psi(y) - D\Psi(x)\Big\rangle_{X\times X^*} \;\;\geq\;\; (\;>\;) \;\;\; 0\,, \tag{2.3}$$

(remember that $D\Psi(x) \in X^*$). Furthermore, Ψ is weakly lower semicontinuous on X. Moreover, if some function $\Psi(x) : X \to \mathbb{R}$ is Gateaux differentiable at every $x \in X$ and satisfy either (2.2) or (2.3) for every $x,y \in X$ s.t. $x \neq y$, then $\Psi(y)$ is convex (strictly convex).

Definition 2.6. Let Z be a Banach space and Z^* be a corresponding dual space. We say that the mapping $\Lambda(z) : Z \to Z^*$ is monotone (strictly monotone) if we have

$$\Big\langle y - z, \Lambda(y) - \Lambda(z)\Big\rangle_{Z\times Z^*} \;\geq\; (>)\, 0 \quad \forall\, y \neq z \in Z\,. \tag{2.4}$$

Definition 2.7. Let Z be a Banach space and Z^* be a corresponding dual space. We say that the mapping $\Lambda(z) : Z \to Z^*$ is pseudo-monotone if for every sequence $\{z_n\}_{n=1}^{+\infty} \subset Z$, satisfying

$$z_n \rightharpoonup z \;\text{ weakly in }\; Z \qquad \text{and} \qquad \varlimsup_{n\to+\infty} \Big\langle z_n - z, \Lambda(z_n)\Big\rangle_{Z\times Z^*} \leq 0 \tag{2.5}$$

we have

$$\varliminf_{n\to+\infty} \Big\langle z_n - y, \Lambda(z_n)\Big\rangle_{Z\times Z^*} \geq \Big\langle z - y, \Lambda(z)\Big\rangle_{Z\times Z^*} \qquad \forall y \in Z\,. \tag{2.6}$$

Lemma 2.2. *Let Z be a Banach space and Z^* be a corresponding dual space. Then the mapping $\Lambda(z) : Z \to Z^*$ is pseudo-monotone if and only if it satisfies the following conditions:*

(i) *For every sequence $\{z_n\}_{n=1}^{+\infty} \subset Z$, such that $z_n \rightharpoonup z$ weakly in Z we have*

$$\underline{\lim}_{n\to+\infty} \Big\langle z_n - z, \Lambda(z_n)\Big\rangle_{Z\times Z^*} \geq 0\,. \tag{2.7}$$

(ii) *If for some sequence $\{z_n\}_{n=1}^{+\infty} \subset Z$, such that $z_n \rightharpoonup z$ weakly in Z we have*

$$\lim_{n\to+\infty} \Big\langle z_n - z, \Lambda(z_n)\Big\rangle_{Z\times Z^*} = 0\,, \tag{2.8}$$

then $\Lambda(z_n) \rightharpoonup \Lambda(z)$ weakly in Z^*.*

Lemma 2.3. *Let Z be a Banach space and Z^* be a corresponding dual space. Assume that the mapping $\Lambda(z) : Z \to Z^*$ is monotone. Moreover assume that $\Lambda(z) : Z \to Z^*$ is continuous for every $z \in Z$ or more generally the function $\zeta_{z,h}(t) : \mathbb{R} \to \mathbb{R}$, defined by*

$$\zeta_{z,h}(t) := \Big\langle h, \Lambda\big(z - th\big)\Big\rangle_{Z\times Z^*} \quad \forall z, h \in Z\,, \quad \forall t \in \mathbb{R}\,, \tag{2.9}$$

is continuous on t for every $z, h \in Z$. Then the mapping $\Lambda(z)$ is pseudo-monotone.

Lemma 2.4. *Let Y and Z be two reflexive Banach spaces. Furthermore, let $S \in \mathcal{L}(Y;Z)$ be an injective operator (i.e. it satisfies* $\ker S = \{0\}$*) and let $S^* \in \mathcal{L}(Z^*;Y^*)$ be the corresponding adjoint operator, which satisfies*

$$\big\langle y, S^*\cdot z^*\big\rangle_{Y\times Y^*} := \big\langle S\cdot y, z^*\big\rangle_{Z\times Z^*} \qquad \text{for every } z^* \in Z^* \text{ and } y \in Y\,. \tag{2.10}$$

Next assume that $a, b \in \mathbb{R}$ s.t. $a < b$. Let $w(t) \in L^\infty(a,b;Y)$ be such that the function $v : [a,b] \to Z$ defined by $v(t) := S\cdot\big(w(t)\big)$ belongs to $W^{1,q}(a,b;Z)$ for some $q \geq 1$. Then we can redefine w on a subset of $[a,b]$ of Lebesgue measure zero, so that $w(t)$ will be Y-weakly continuous in t on $[a,b]$ (i.e. $w \in C^0_w(a,b;Y)$). Moreover, for every $a \leq \alpha < \beta \leq b$ and for every $\delta(t) \in C^1\big([a,b];Z^*\big)$ we will have

$$\int\limits_\alpha^\beta \bigg\{\Big\langle \frac{dv}{dt}(t), \delta(t)\Big\rangle_{Z\times Z^*} + \Big\langle v(t), \frac{d\delta}{dt}(t)\Big\rangle_{Z\times Z^*}\bigg\}dt = \big\langle w(\beta), S^*\cdot\delta(\beta)\big\rangle_{Y\times Y^*} - \big\langle w(\alpha), S^*\cdot\delta(\alpha)\big\rangle_{Y\times Y^*}\,. \tag{2.11}$$

Definition 2.8. Let X be a reflexive Banach space and X^* the corresponding dual space. Furthermore let H be a Hilbert space and $T \in \mathcal{L}(X,H)$ be an injective (i.e. it satisfies $\ker T = \{0\}$) inclusion operator such that its image is dense on H. Then we call the triple $\{X, H, X^*\}$ an evolution triple with the corresponding inclusion operator T. Throughout this paper we assume the space H^* be equal to H (remember that H is a Hilbert space) but in general we don't associate X^* with X even in the case where X is a Hilbert space (and thus X^* will be isomorphic to X). Further we define the bounded linear operator $\widetilde{T} \in \mathcal{L}(H;X^*)$ by the formula

$$\big\langle x, \widetilde{T}\cdot y\big\rangle_{X\times X^*} := \big\langle T\cdot x, y\big\rangle_{H\times H} \qquad \text{for every } y \in H \text{ and } x \in X\,. \tag{2.12}$$

In particular $\|\widetilde{T}\|_{\mathcal{L}(H;X^*)} = \|T\|_{\mathcal{L}(X;H)}$ and since we assumed that the image of T is dense in H we deduce that $\ker\widetilde{T} = \{0\}$ and so $\widetilde{T}$ is an injective operator. So $\widetilde{T}$ is an inclusion of H to X^* and the operator $I := \widetilde{T}\circ T$ is an injective inclusion of X to X^*. Furthermore, clearly

$$\big\langle x, I\cdot z\big\rangle_{X\times X^*} = \big\langle T\cdot x, T\cdot z\big\rangle_{H\times H} = \big\langle z, I\cdot x\big\rangle_{X\times X^*} \qquad \text{for every } x, z \in X\,. \tag{2.13}$$

So $I \in \mathcal{L}(X, X^*)$ is self-adjoint operator. Moreover, I is strictly positive, since

$$\big\langle x, I\cdot x\big\rangle_{X\times X^*} = \|T\cdot x\|_H^2 > 0 \qquad \forall x \neq 0 \in X\,. \tag{2.14}$$

Lemma 2.5. *Let X be a reflexive Banach space and X^* the corresponding dual space. Furthermore let $I \in \mathcal{L}(X, X^*)$ be a self-adjoint and strictly positive operator. i.e.*

$$\big\langle x, I \cdot z \big\rangle_{X \times X^*} = \big\langle z, I \cdot x \big\rangle_{X \times X^*} \qquad \text{for every } x, z \in X\,, \tag{2.15}$$

and

$$\big\langle x, I \cdot x \big\rangle_{X \times X^*} > 0 \qquad \forall x \neq 0 \in X\,. \tag{2.16}$$

Then there exists a Hilbert space H and an injective operator $T \in \mathcal{L}(X, H)$ (i.e. $\ker T = \{0\}$), whose image is dense in H, and such that if we consider the operator $\widetilde{T} \in \mathcal{L}(H; X^*)$, defined by the formula (2.12)*, then we will have*

$$(\widetilde{T} \circ T) \cdot x = I \cdot x \qquad \forall x \in X\,. \tag{2.17}$$

I.e. $\{X, H, X^*\}$ is an evolution triple with the corresponding inclusion operator $T \in \mathcal{L}(X; H)$, as it was defined in Definition 2.8, together with the corresponding operator $\widetilde{T} \in \mathcal{L}(H; X^*)$, defined as in (2.12)*, and $I \equiv \widetilde{T} \circ T$.*

Next as a particular case of Lemma 2.4 we have the following Corollary.

Corollary 2.1. *Let $\{X, H, X^*\}$ be an evolution triple with the corresponding inclusion operator $T \in \mathcal{L}(X; H)$ as it was defined in Definition 2.8 together with the corresponding operator $\widetilde{T} \in \mathcal{L}(H; X^*)$ defined as in* (2.12) *and let $a, b \in \mathbb{R}$ s.t. $a < b$. Let $w(t) \in L^\infty(a, b; H)$ be such that the function $v : [a, b] \to X^*$ defined by $v(t) := \widetilde{T} \cdot \big(w(t)\big)$ belongs to $W^{1,q}(a, b; X^*)$ for some $q \geq 1$. Then we can redefine w on a subset of $[a, b]$ of Lebesgue measure zero, so that $w(t)$ will be H-weakly continuous in t on $[a, b]$ (i.e. $w \in C^0_w(a, b; H)$). Moreover, for every $a \leq \alpha < \beta \leq b$ and for every $\delta(t) \in C^1\big([a, b]; X\big)$ we will have*

$$\int\limits_\alpha^\beta \bigg\{ \Big\langle \delta(t), \frac{dv}{dt}(t) \Big\rangle_{X \times X^*} + \Big\langle \frac{d\delta}{dt}(t), v(t) \Big\rangle_{X \times X^*} \bigg\} dt = \big\langle T \cdot \delta(\beta), w(\beta) \big\rangle_{H \times H} - \big\langle T \cdot \delta(\alpha), w(\alpha) \big\rangle_{H \times H}\,. \tag{2.18}$$

Lemma 2.6. *Let $\{X, H, X^*\}$ be an evolution triple with the corresponding inclusion operator $T \in \mathcal{L}(X; H)$ as it was defined in Definition 2.8 together with the corresponding operator $\widetilde{T} \in \mathcal{L}(H; X^*)$ defined as in* (2.12) *and let $a, b \in \mathbb{R}$ s.t. $a < b$. Let $u(t) \in L^q(a, b; X)$ for some $q > 1$ such that the function $v(t) : [a, b] \to X^*$ defined by $v(t) := I \cdot \big(u(t)\big)$ belongs to $W^{1,q^*}(a, b; X^*)$ for $q^* := q/(q-1)$, where we denote $I := \widetilde{T} \circ T : X \to X^*$. Then the function $w(t) : [a, b] \to H$ defined by $w(t) := T \cdot \big(u(t)\big)$ belongs to $L^\infty(a, b; H)$ and for every subinterval $[\alpha, \beta] \subset [a, b]$ we have*

$$\int_\alpha^\beta \Big\langle u(t), \frac{dv}{dt}(t) \Big\rangle_{X \times X^*} dt = \frac{1}{2}\Big(\|w(\beta)\|_H^2 - \|w(\alpha)\|_H^2 \Big)\,, \tag{2.19}$$

up to a redefinition of $w(t)$ on a subset of $[a, b]$ of Lebesgue measure zero, such that w is H-weakly continuous, as it was stated in Corollary 2.1.

We will need in the sequel the following compactness results.

Lemma 2.7. *Let X, Y Z be three Banach spaces, such that X is a reflexive space. Furthermore, let $T \in \mathcal{L}(X; Y)$ and $S \in \mathcal{L}(X; Z)$ be bounded linear operators. Moreover assume that S is an injective inclusion (i.e. it satisfies $\ker S = \{0\}$) and T is a compact operator. Assume that $a, b \in \mathbb{R}$ such that $a < b$, $1 \leq q < +\infty$ and $\{u_n(t)\} \subset L^q(a, b; X)$ is a bounded in $L^q(a, b; X)$ sequence of functions, such that the functions $v_n(t) : (a, b) \to Z$, defined by $v_n(t) := S \cdot \big(u_n(t)\big)$, belongs to $L^\infty(a, b; Z)$, the sequence $\{v_n(t)\}$ is bounded in $L^\infty(a, b; Z)$ and for a.e. $t \in (a, b)$ we have*

$$v_n(t) \rightharpoonup v(t) \quad \text{weakly in } Z \text{ as } n \to +\infty\,. \tag{2.20}$$

Then,

$$\big\{T \cdot \big(u_n(t)\big)\big\} \quad \text{converges strongly in } L^q(a, b; Y)\,. \tag{2.21}$$

Lemma 2.8. *Let Z be a reflexive Banach space and let $\{v_n(t)\}_{n=1}^{+\infty} \subset W^{1,1}(a,b;Z)$ be a sequence of functions, bounded in $W^{1,1}(a,b;Z)$. Then, $\{v_n(t)\}_{n=1}^{+\infty}$ is bounded in $L^\infty(a,b;Z)$ and, up to a subsequence, we have*

$$v_n(t) \rightharpoonup v(t) \quad \text{weakly in } Z \text{ as } n \to +\infty\,, \quad \text{for a.e } t \in (a,b)\,. \tag{2.22}$$

As a direct consequence of Lemma 2.7 and Lemma 2.8 we have the following Lemma.

Lemma 2.9. *Let X, Y and Z be three Banach spaces, such that X and Z are reflexive. Furthermore, let $T \in \mathcal{L}(X;Y)$ and $S \in \mathcal{L}(X;Z)$ be bounded linear operators. Moreover assume that S is an injective inclusion (i.e. it satisfies $\ker S = \{0\}$) and T is a compact operator. Assume that $a, b \in \mathbb{R}$ such that $a < b$, $1 \le q < +\infty$ and $\{u_n(t)\} \subset L^q(a,b;X)$ is a bounded in $L^q(a,b;X)$ sequence of functions, such that the functions $v_n(t) : (a,b) \to Z$, defined by $v_n(t) := S \cdot \big(u_n(t)\big)$, belongs to $W^{1,1}(a,b;Z)$ and the sequence $\left\{\frac{dv_n}{dt}(t)\right\}$ is bounded in $L^1(a,b;Z)$. Then, up to a subsequence,*

$$\big\{T \cdot \big(u_n(t)\big)\big\} \quad \text{converges strongly in } L^q(a,b;Y)\,. \tag{2.23}$$

The following simple embedding result was proven in the Appendix of [15]:

Lemma 2.10. *Let X be a separable Banach space. Then there exists a separable Hilbert space Y and a bounded linear inclusion operator $S \in \mathcal{L}(Y;X)$ such that S is injective (i.e. $\ker S = \{0\}$), the image of S is dense in X and moreover, S is a compact operator.*

In the future we also need the following simple Lemma:

Lemma 2.11. *Let X be a reflexive Banach space and let $\Psi(x) : X \to [0,+\infty)$ be a convex function which is Gateaux differentiable on every $x \in X$, satisfies $\Psi(0) = 0$ and satisfies*

$$0 \le \Psi(x) \le C_0 \|x\|_X^q + C_0 \quad \forall x \in X\,, \tag{2.24}$$

for some $q > 1$ and $C_0 > 0$. Then for some $\bar{C}_0$, that depends only on C_0 and q from (2.24), we have

$$\|D\Psi(x)\|_{X^*} \le \bar{C}_0 \|x\|_X^{q-1} + \bar{C}_0 \quad \forall x \in X\,. \tag{2.25}$$

Proof. Since Ψ is convex, from (2.2), for every $x, h \in X$ we have

$$\big\langle h, D\Psi(x)\big\rangle_{X \times X^*} \le \Psi(x+h) - \Psi(x)\,. \tag{2.26}$$

Therefore, for every $x, h \in X$ such that $\|h\|_X \le 1$ and $\|x\|_X \ge 1$ we have

$$\big\langle h, D\Psi(x)\big\rangle_{X \times X^*} \le \frac{1}{\|x\|_X}\Big(\Psi\big(x + \|x\|_X h\big) - \Psi(x)\Big)\,. \tag{2.27}$$

Thus using growth condition (2.24) we deduce that for every $x, h \in X$ such that $\|h\|_X \le 1$ and $\|x\|_X \ge 1$ we have

$$\big\langle h, D\Psi(x)\big\rangle_{X \times X^*} \le \tilde{C} \|x\|_X^{q-1}\,, \tag{2.28}$$

and so

$$\|D\Psi(x)\|_{X^*} \le \tilde{C} \|x\|_X^{q-1}\,, \tag{2.29}$$

for every x which satisfy $\|x\|_X \ge 1$. However, by (2.26) and (2.24) we have

$$\big\langle h, D\Psi(x)\big\rangle_{X \times X^*} \le \hat{C}\,, \tag{2.30}$$

for every $x, h \in X$ such that $\|x\|_X \le 1$ and $\|h\|_X \le 1$, where $\hat{C} > 0$ is a constant. So $\|D\Psi(x)\|_{X^*} \le \hat{C}$ for every x which satisfy $\|x\|_X \le 1$. This together with (2.29) gives the desired result (2.25). $\square$

3 The Existence results

Definition 3.1. Let $\{X,H,X^*\}$ be an evolution triple with the corresponding inclusion operator $T\in\mathcal{L}(X;H)$ as it was defined in Definition 2.8. Furthermore let (a,b) be a real interval, $q>1$ and $q^*:=q/(q-1)$. We say that the mapping $\Gamma(u):\big\{u\in L^q(a,b;X):\ T\cdot u\in L^\infty(a,b;H)\big\}\to L^{q^*}(a,b;X^*)\equiv\big\{L^q(a,b;X)\big\}^*$ is weakly pseudo-monotone if for every sequence $\big\{u_n(t)\big\}_{n=1}^{+\infty}\subset L^q(a,b;X)$, such that $\big\{T\cdot u_n(t)\big\}_{n=1}^{+\infty}\subset L^\infty(a,b;H)$, $u_n(t)\rightharpoonup u(t)$ weakly in $L^q(a,b;X)$, $\big\{T\cdot u_n(t)\big\}_{n=1}^{+\infty}$ is bounded in $L^\infty(a,b;H)$ and $T\cdot u_n(t)\rightharpoonup T\cdot u(t)$ weakly in H for a.e. $t\in(a,b)$ the following conditions are satisfied.

- $$\underline{\lim}_{n\to+\infty}\Big\langle u_n-u,\Gamma(u_n)\Big\rangle_{L^q(a,b;X)\times L^{q^*}(a,b;X^*)}\geq 0\,. \tag{3.1}$$
- If we have $$\lim_{n\to+\infty}\Big\langle u_n-u,\Gamma(u_n)\Big\rangle_{L^q(a,b;X)\times L^{q^*}(a,b;X^*)}=0\,, \tag{3.2}$$ then $\Gamma(u_n)\rightharpoonup\Gamma(u)$ weakly in $L^{q^*}(a,b;X^*)$.

Remark 3.1. By Lemmas 2.2 and 2.3 we know that if the mapping $\Gamma(u):L^q(a,b;X)\to L^{q^*}(a,b;X^*)$ is pseudo-monotone then $\Gamma(u)$ is weakly pseudo-monotone. Moreover, by Lemma 2.3 we infer that if we assume $\Gamma(u):\big\{u\in L^q(a,b;X):\ T\cdot u\in L^\infty(a,b;H)\big\}\to L^{q^*}(a,b;X^*)$ to be monotone, the function

$$f_{u,h}(s):=\Big\langle h,\Gamma(u-sh)\Big\rangle_{L^q(a,b;X)\times L^{q^*}(a,b;X^*)}\quad\forall u,h\in\big\{\zeta\in L^q(a,b;X):\ T\cdot\zeta\in L^\infty(a,b;H)\big\},\ \forall s\in\mathbb{R},$$

to be continuous on s, and $\Gamma(u)$ to be locally bounded on $\big\{u\in L^q(a,b;X):\ T\cdot u\in L^\infty(a,b;H)\big\}$ (i.e. for every sequence $\{u_n(t)\}$ bounded in $L^q(a,b;X)$, such that $\{T\cdot u_n(t)\}$ is bounded in $L^\infty(a,b;H)$, $\Gamma(u_n)$ is assumed to be bounded in $L^{q^*}(a,b;X^*)$), then the mapping $\Gamma(u)$ is weakly pseudo-monotone.

Remark 3.2. It is trivially follows from the definition that if $\Gamma_1(u),\Gamma_2(u):\big\{u\in L^q(a,b;X):\ T\cdot u\in L^\infty(a,b;H)\big\}\to L^{q^*}(a,b;X^*)$ are two weakly pseudo-monotone mappings, then the sum of them, $\Gamma_1(u)+\Gamma_2(u)$ is also a weakly pseudo-monotone mapping.

Definition 3.2. Let $\{X,H,X^*\}$ be an evolution triple with the corresponding inclusion operator $T\in\mathcal{L}(X;H)$ as it was defined in Definition 2.8 together with the corresponding operator $\widetilde{T}\in\mathcal{L}(H;X^*)$ defined as in (2.12). Furthermore, let Y be a reflexive Banach space and $S\in\mathcal{L}(Y,X)$ be an injective operator such that its image is dense in X. Moreover, assume that S is a compact operator and let $S^*\in\mathcal{L}(X^*;Y^*)$ be the corresponding adjoint operator, which satisfy

$$\big\langle y,S^*\cdot x^*\big\rangle_{Y\times Y^*}:=\big\langle S\cdot y,x^*\big\rangle_{X\times X^*}\qquad\text{for every } x^*\in X^*\text{ and } y\in Y\,. \tag{3.3}$$

Set $P\in\mathcal{L}(Y;H)$, defined by $P:=T\circ S$ and $\widetilde{P}\in\mathcal{L}(H;Y^*)$, defined by $\widetilde{P}:=S^*\circ\widetilde{T}$. Then it is clear that $\{Y,H,Y^*\}$ is another evolution triple with the corresponding inclusion operator $P\in\mathcal{L}(Y;H)$ as it was defined in Definition 2.8 together with the corresponding adjoint operator $\widetilde{P}\in\mathcal{L}(H;Y^*)$ defined as in (2.12). We will call the quintette $\{Y,X,H,X^*,Y^*\}$ together with the corresponding operators $S\in\mathcal{L}(Y,X)$, $T\in\mathcal{L}(X;H)$, $\widetilde{T}\in\mathcal{L}(H;X^*)$ and $S^*\in\mathcal{L}(X^*;Y^*)$ an evolution quintette. Furthermore consider $a,b\in\mathbb{R}$ s.t. $a<b$. Let $\psi(t)\in L^q(a,b;Y)$ for some $q>1$ such that the function $\varphi(t):(a,b)\to X^*$ defined by $\varphi(t):=I_Y\cdot\big(\psi(t)\big)$ belongs to $W^{1,q^*}(a,b;Y^*)$ for $q^*:=q/(q-1)$, where $I_Y:=\widetilde{P}\circ P:Y\to Y^*$. Denote the set of all such functions ψ by $\mathcal{R}_{Y,q}(a,b)$. As before, by Lemma 2.6, for every $\psi(t)\in\mathcal{R}_q(a,b)$ the function $w(t):[a,b]\to H$ defined by $w(t):=P\cdot\big(\psi(t)\big)$ belongs to $L^\infty(a,b;H)$ and, up to a redefinition of $w(t)$ on a subset of $[a,b]$ of Lebesgue measure zero, w is H-weakly continuous, as it was stated in Corollary 2.1.

Definition 3.3. Let $\{Y,X,H,X^*,Y^*\}$ be an evolution quintette with the corresponding inclusion operators $S\in\mathcal{L}(Y,X)$, $T\in\mathcal{L}(X;H)$, $\widetilde{T}\in\mathcal{L}(H;X^*)$ and $S^*\in\mathcal{L}(X^*;Y^*)$ as it was defined in Definition 3.2 together with the corresponding operators $P\in\mathcal{L}(Y;H)$ and $\widetilde{P}\in\mathcal{L}(H;Y^*)$ defined by

$P := T \circ S$ and $\widetilde{P} := S^* \circ \widetilde{T}$ (Remember that the operator S is compact by definition). Furthermore, let $a, b, q \in \mathbb{R}$ s.t. $a < b$, $q \geq 2$ and $q := q/(q-1)$. Next let $\Psi(y) : Y \to [0, +\infty)$ be a convex function which is Gateaux differentiable on every $y \in Y$, satisfies $\Psi(0) = 0$ and satisfies the growth condition

$$(1/C_0)\,\|y\|_Y^q - C_0 \leq \Psi(y) \leq C_0\,\|y\|_Y^q + C_0 \quad \forall y \in Y\,, \tag{3.4}$$

and uniform convexity condition

$$\Big\langle y_1 - y_2, D\Psi(y_1) - D\Psi(y_2)\Big\rangle_{Y \times Y^*} \geq (1/C_0)\,\|y_1 - y_2\|_Y^2 \quad \forall y_1, y_2 \in Y\,, \tag{3.5}$$

for some $C_0 > 0$. Next, for every $t \in [a, b]$ let $\Phi_t(x) : X \to [0, +\infty)$ be a convex function which is Gateaux differentiable at every $x \in X$, satisfies $\Phi_t(0) = 0$ and satisfies the growth condition

$$0 \leq \Phi_t(x) \leq C\,\|x\|_X^q + C \quad \forall x \in X,\ \forall t \in [a, b]\,, \tag{3.6}$$

for some $C > 0$. We also assume that $\Phi_t(x)$ is Borel on the pair of variables (x, t). Furthermore, for every $t \in [a, b]$ let $\Lambda_t(x) : X \to X^*$ be a function which is Gateaux differentiable at every $x \in X$, $\Lambda_t(0) \in L^{q^*}(a, b; X^*)$ and the derivative of Λ_t satisfies the growth condition

$$\|D\Lambda_t(x)\|_{\mathcal{L}(X;X^*)} \leq g\big(\|T \cdot x\|_H\big)\left(\|x\|_X^{q-2} + 1\right) \quad \forall x \in X,\ \forall t \in [a, b]\,, \tag{3.7}$$

for some nondecreasing function $g(s) : [0, +\infty) \to (0, +\infty)$. We also assume that $\Lambda_t(x)$ is strongly Borel on the pair of variables (x, t) (see Definition 2.2). Assume also that Λ_t and Φ_t satisfy the following monotonicity and positivity conditions

$$\begin{aligned}&\Big\langle h, \big\{D\Phi_t(x+h) - D\Phi_t(x)\big\} + D\Lambda_t(x) \cdot h\Big\rangle_{X \times X^*} \geq \\ &- \hat{g}\big(\|T \cdot x\|_H\big) \cdot \Big(\|x\|_X^q + \mu(t)\Big)^{(2-p)/2} \cdot \|h\|_X^p \cdot \|T \cdot h\|_H^{(2-p)} \qquad \forall x \in X, \forall h \in X\ \forall t \in [a, b]\,,\end{aligned} \tag{3.8}$$

and

$$\Big\langle x, D\Phi_t(x) + \Lambda_t(x)\Big\rangle_{X \times X^*} \geq \frac{k_0}{\hat{C}}\,\|x\|_X^q - \hat{C}\Big(\|x\|_X^p + 1\Big)\Big(\|T \cdot x\|_H^{(2-p)} + 1\Big) - \mu(t) \quad \forall x \in X,\ \forall t \in [a, b], \tag{3.9}$$

where $p \in [0, 2)$, $k_0 \in \{0, 1\}$ and $\hat{C} > 0$ are some constants, $\hat{g}(s) : [0, +\infty) \to (0, +\infty)$ is some nondecreasing function and $\mu(t) \in L^1\big(a, b; \mathbb{R}\big)$ is some nonnegative function.

Lemma 3.1. *Assume that all the conditions of Definition 3.2 and Definition 3.3 are satisfied. Moreover, let $w^{(0)} = P \cdot \psi^{(0)}$ where $\psi^{(0)} \in Y$. Then for every $\varepsilon > 0$ there exists $\psi(t) \in \mathcal{R}_{Y,q}(a, b)$, such that $\psi(t)$ is a solution to*

$$\frac{d\varphi}{dt}(t) + S^* \cdot \Big(\Lambda_t\big(u(t)\big) + D\Phi_t\big(u(t)\big)\Big) + \varepsilon D\Psi\big(\psi(t)\big) = 0 \ \text{ for a.e. } t \in (a, b)\,, \quad \text{and } w(a) = w^{(0)}\,, \tag{3.10}$$

where $u(t) := S \cdot \big(\psi(t)\big)$, $w(t) := P \cdot \big(\psi(t)\big)$, $\varphi(t) := I_Y \cdot \big(\psi(t)\big) = \widetilde{P} \cdot \big(w(t)\big)$ with $I_Y := \widetilde{P} \circ P : Y \to Y^*$ and we assume that $w(t)$ is H-weakly continuous on $[a, b]$, as it was stated in Corollary 2.1

Proof. The result follows by applying Corollary 1.1 (see also Proposition 3.2 from [15]) with Y and Y^* instead of X and X^*, P and $\tilde{P}$ instead of T and $\tilde{T}$, ψ instead of u, u instead of $S \cdot u$, φ instead of v, $\varepsilon\Psi(\psi) + \Phi_t(S \cdot \psi)$ instead of $\Psi_t(u)$ and $S^* \cdot \big(\Lambda_t(S \cdot \psi)\big)$ instead of $A_t(S \cdot u)$. □

Theorem 3.1. *Suppose that all the conditions of Definition 3.2 and Definition 3.3 are satisfied together with the assumption $k_0 = 1$ in (3.9). Moreover, assume that the mapping $\Gamma\big(x(t)\big) : \big\{\bar{x}(t) \in L^q(a, b; X) :\ T \cdot \bar{x}(t) \in L^\infty(a, b; H)\big\} \to L^{q^*}(a, b; X^*)$, defined by*

$$\begin{aligned}\Big\langle h(t), \Gamma\big(x(t)\big)\Big\rangle_{L^q(a,b;X) \times L^{q^*}(a,b;X^*)} &:= \int_a^b \Big\langle h(t), \Lambda_t\big(x(t)\big)\Big\rangle_{X \times X^*} dt \\ &\forall\, x(t) \in \big\{\bar{x}(t) \in L^q(a, b; X) :\ T \cdot \bar{x}(t) \in L^\infty(a, b; H)\big\},\ \forall\, h(t) \in L^q(a, b; X)\,,\end{aligned} \tag{3.11}$$

is weakly pseudo-monotone with respect to the evolution triple $\{X, H, X^*\}$ (see Definition 3.1). Furthermore, let $w_n^{(0)} = P \cdot \psi_n^{(0)}$ where $\{\psi_n^{(0)}\}_{n=1}^{\infty} \subset Y$ be such that $w_n^{(0)} \to w_0$ strongly in H and let $\varepsilon_n > 0$ be such that $\varepsilon_n \to 0$ as $n \to +\infty$. Moreover, assume that $\psi_n(t) \in \mathcal{R}_{Y,q}(a,b)$ is a solution to

$$\frac{d\varphi_n}{dt}(t)+S^*\cdot\Big(\Lambda_t\big(u_n(t)\big)+D\Phi_t\big(u_n(t)\big)\Big)+\varepsilon_n D\Psi\big(\psi_n(t)\big) = 0 \;\; \textit{for a.e. } t \in (a,b) \quad \textit{and} \;\; w_n(a) = w_n^{(0)}, \tag{3.12}$$

where $u_n(t) := S \cdot \big(\psi_n(t)\big)$, $w_n(t) := P \cdot \big(\psi_n(t)\big)$, $\varphi_n(t) := \widetilde{P} \cdot \big(w_n(t)\big)$ and we assume that $w_n(t)$ is H-weakly continuous on $[a,b]$, as it was stated in Corollary 2.1 (As we saw in Lemma 3.1 such a solution exists). Then, up to a subsequence, we have $u_n(t) \rightharpoonup u(t)$ weakly in $L^q(a,b;X)$ where $u(t) \in L^q(a,b;X)$ is such that $w(t) := T \cdot \big(u(t)\big) \in L^\infty(a,b;H)$, $v(t) := \widetilde{T} \cdot \big(w(t)\big) = \widetilde{T} \circ T\big(u(t)\big) \in W^{1,q^*}(a,b;X^*)$ and $u(t)$ is a solution to

$$\begin{cases} \frac{dv}{dt}(t) + \Lambda_t\big(u(t)\big) + D\Phi_t\big(u(t)\big) = 0 \quad \textit{for a.e. } t \in (a,b)\,, \\ w(a) = w_0\,, \end{cases} \tag{3.13}$$

where we assume that $w(t)$ is H-weakly continuous on $[a,b]$, as it was stated in Corollary 2.1. So in the conditions of the theorem a solution to (3.13) exists for every $w_0 \in H$.

Proof. By (3.12) we deduce

$$\int_a^t \Big\langle \psi_n(s), \frac{d\varphi_n}{dt}(s) \Big\rangle_{Y\times Y^*} ds + \int_a^t \Big\langle u_n(s), \Lambda_s\big(u_n(s)\big) + D\Phi_s\big(u_n(s)\big) \Big\rangle_{X\times X^*} ds \\ + \varepsilon_n \int_a^t \Big\langle \psi_n(s), D\Psi\big(\psi_n(s)\big) \Big\rangle_{Y\times Y^*} ds = 0 \qquad \forall t \in [a,b]\,. \tag{3.14}$$

Next, since by Lemma 2.6 we have

$$\int_a^t \Big\langle \psi_n(s), \frac{d\varphi_n}{dt}(s) \Big\rangle_{Y\times Y^*} ds = \frac{1}{2}\Big(\big\|w_n(t)\big\|_H^2 - \big\|w_n^{(0)}\big\|_H^2\Big)\,,$$

using (3.14) we obtain

$$\frac{1}{2}\big\|w_n(t)\big\|_H^2 + \int_a^t \Big\langle u_n(s), \Lambda_s\big(u_n(s)\big) + D\Phi_s\big(u_n(s)\big) \Big\rangle_{X\times X^*} ds \\ + \varepsilon_n \int_a^t \Big\langle \psi_n(s), D\Psi\big(\psi_n(s)\big) \Big\rangle_{Y\times Y^*} ds = \frac{1}{2}\big\|w_n^{(0)}\big\|_H^2 \qquad \forall t \in [a,b]\,. \tag{3.15}$$

However, since $\Psi(\cdot)$ is convex and since $\Psi(\cdot) \geq 0$, $\Psi(0) = 0$ and then also $D\Psi(0) = 0$, by (2.2) we have

$$\Big\langle \psi_n(t), D\Psi\big(\psi_n(t)\big) \Big\rangle_{Y\times Y^*} \geq \Psi\big(\psi_n(t)\big) \geq 0 \qquad \forall t \in (a,b)\,. \tag{3.16}$$

In the same way

$$\Big\langle u_n(t), D\Phi_t\big(u_n(t)\big) \Big\rangle_{X\times X^*} \geq \Phi_t\big(u_n(t)\big) \geq 0 \qquad \forall t \in (a,b)\,. \tag{3.17}$$

Therefore, from (3.15) we deduce

$$\frac{1}{2}\big\|w_n(t)\big\|_H^2 + \int_a^t \Big\langle u_n(s), \Lambda_s\big(u_n(s)\big) + D\Phi_s\big(u_n(s)\big) \Big\rangle_{X\times X^*} ds \leq \frac{1}{2}\big\|w_n^{(0)}\big\|_H^2 \qquad \forall t \in [a,b]\,. \tag{3.18}$$

On the other hand by (3.9) we infer

$$\int_a^t \Big\langle u_n(s), D\Phi_s\big(u_n(s)\big) + \Lambda_s\big(u_n(s)\big) \Big\rangle_{X\times X^*} ds \geq \\ \int_a^t \Big\{ \frac{1}{C}\|u_n(s)\|_X^q - C\Big(\|u_n(s)\|_X^p + 1\Big) \cdot \Big(\|w_n(s)\|^{(2-p)} + 1\Big) \Big\} ds - C \quad \forall t \in (a,b)\,. \tag{3.19}$$

Therefore, inserting (3.19) into (3.18) we deduce that there exists $C_1 > 0$ such that

$$\big\|w_n(t)\big\|_H^2 + \int_a^t \big\|u_n(s)\big\|_X^q ds \leq C_1 + C_1 \int_a^t \big\|u_n(s)\big\|_X^p \cdot \big\|w_n(s)\big\|_H^{(2-p)} ds \leq \\ C_1 + C_1 \bigg(\int_a^t \big\|u_n(s)\big\|_X^2 ds\bigg)^{p/2} \cdot \bigg(\int_a^t \big\|w_n(s)\big\|_H^2 ds\bigg)^{(2-p)/2} \qquad \forall t \in [a,b] \,. \quad (3.20)$$

Thus, since $q \geq 2$ and $p \in [0,2)$, in particular

$$\int_a^t \big\|u_n(s)\big\|_X^2 ds \leq C_2 + C_2 \cdot \int_a^t \big\|w_n(s)\big\|_H^2 ds \qquad \forall t \in [a,b] \,, \tag{3.21}$$

and then from (3.20) we deduce

$$\big\|w_n(t)\big\|_H^2 + \int_a^t \big\|u_n(s)\big\|_X^q ds \leq C_3 + C_3 \int_a^t \big\|w_n(s)\big\|_H^2 ds \qquad \forall t \in [a,b] \,. \tag{3.22}$$

In particular we deduce

$$\|w_n(t)\|_H^2 \leq C_4 \int_a^t \|w_n(s)\|_H^2 ds + C_4 \qquad \forall t \in [a,b] \quad \forall n \in \mathbb{N} \,, \tag{3.23}$$

where $C_4 > 0$ doesn't depend on n and t. Then

$$\frac{d}{dt}\bigg\{\int_a^t \|w_n(s)\|_H^2 ds \cdot \exp\left(-C_4 t\right)\bigg\} \leq C_4 \exp\left(-C_4 t\right) \qquad \forall t \in [a,b] \quad \forall n \in \mathbb{N} \,, \tag{3.24}$$

and thus

$$\int_a^t \|w_n(s)\|_H^2 ds \leq \exp\left\{C_4(t-a)\right\} - 1 \leq \exp\left\{C_4(b-a)\right\} \qquad \forall t \in [a,b] \quad \forall n \in \mathbb{N} \,. \tag{3.25}$$

Therefore, by (3.23) the sequence $\{w_n(t)\}$ is bounded in $L^\infty(a,b;H)$. Thus by (3.22) we also obtain that the sequence $\{u_n(t)\}$ is bounded in $L^q(a,b;X)$. So

$$\begin{cases} \{u_n(t)\} & \text{is bounded in } \ L^q(a,b;X) \,, \\ \{w_n(t)\} & \text{is bounded in } \ L^\infty(a,b;H) \,. \end{cases} \tag{3.26}$$

Therefore, since $L^q(a,b;X)$ is a reflexive space, up to a subsequence we have

$$\begin{cases} u_n(t) \rightharpoonup u(t) & \text{weakly in } \ L^q(a,b;X) \,, \\ w_n(t) \rightharpoonup w(t) & \text{weakly in } \ L^2(a,b;H) \,, \end{cases} \tag{3.27}$$

where $w(t) := T \cdot \big(u(t)\big)$. Next plugging (3.26) into (3.15) and using (3.7) and (3.17) we deduce

$$\varepsilon_n \int_a^b \Big\langle \psi_n(s), D\Psi\big(\psi_n(s)\big)\Big\rangle_{Y \times Y^*} ds \leq \bar{C}_4 \,, \tag{3.28}$$

where $\bar{C}_4$ is a constant. Then using (3.16) and the growth condition (3.4), we deduce from (3.28),

$$\varepsilon_n \int_a^b \big\|\psi_n(s)\big\|_Y^q ds \leq C_5 \,. \tag{3.29}$$

On the other hand by (2.25) in Lemma 2.11, for some $\bar{C} > 0$ we have

$$\Big\|D\Psi\big(\psi_n(t)\big)\Big\|_{Y^*} \leq \bar{C}\big\|\psi_n(t)\big\|_Y^{q-1} + \bar{C} \qquad \forall t \in (a,b) \,, \tag{3.30}$$

and then

$$\Big\|D\Psi\big(\psi_n(t)\big)\Big\|_{Y^*}^{q^*}\leq \bar{C}_0\big\|\psi_n(t)\big\|_{Y}^{q}+\bar{C}_0\qquad\forall t\in(a,b)\,,\tag{3.31}$$

Thus plugging (3.31) into (3.29) we deduce

$$\int_a^b\Big\|\varepsilon_n D\Psi\big(\psi_n(s)\big)\Big\|_{Y^*}^{q^*}ds\leq \hat{C}\varepsilon_n^{(q^*-1)}=\hat{C}\varepsilon_n^{1/(q-1)}\,.\tag{3.32}$$

So,

$$\lim_{n\to+\infty}\Big\|\varepsilon_n D\Psi\big(\psi_n(t)\big)\Big\|_{L^{q^*}(a,b;Y^*)}=0\,.\tag{3.33}$$

In particular, using (3.26), (3.27), (3.33), (3.12), (3.7), (3.6) and the growth condition (2.25) in Lemma 2.11 we deduce that

$$\frac{d\varphi_n}{dt}(t)\quad\text{is bounded in }\;L^{q^*}(a,b;Y^*)\,.\tag{3.34}$$

Next for every $t\in[a,b]$ define $\Theta_t(x):X\to X^*$ by

$$\Theta_t(x):=\Lambda_t(x)+D\Phi_t(x)\qquad\forall x\in X\,.\tag{3.35}$$

Then by the growth condition (2.25) in Lemma 2.11 and by the growth condition (3.7) we obtain

$$\big\|\Theta_t(x)\big\|_{X^*}\leq g\big(\|T\cdot x\|_H\big)\cdot\big(\|x\|_X^{q-1}+1\big)+\big\|\Lambda_t(0)\big\|_{X^*}\qquad\forall x\in X,\;\;\forall t\in(a,b)\,,\tag{3.36}$$

for some constant $C>0$. Thus using (3.26), (3.27) and (3.36) we deduce that, up to a further subsequence,

$$\Theta_t\big(u_n(t)\big)\rightharpoonup\Xi(t)\quad\text{weakly in }\;L^{q^*}(a,b;X^*)\,.\tag{3.37}$$

On the other hand by (3.12) and by Corollary 2.1 for every $\delta(t)\in C^1\big([a,b];Y\big)$ and every $\beta\in[a,b]$ we have

$$\Big\langle T\cdot\big(S\cdot\delta(\beta)\big),w_n(\beta)\Big\rangle_{H\times H}-\Big\langle T\cdot\big(S\cdot\delta(a)\big),w_n^{(0)}\Big\rangle_{H\times H}-\int\limits_a^\beta\Big\langle S\cdot\Big(\frac{d\delta}{dt}(t)\Big),v_n(t)\Big\rangle_{X\times X^*}dt+\\ \int\limits_a^\beta\Big\langle\delta(t),\varepsilon_n D\Psi\big(\psi_n(t)\big)\Big\rangle_{Y\times Y^*}dt+\int\limits_a^\beta\Big\langle\big(S\cdot\delta(t)\big),\Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*}ds=0\,,\tag{3.38}$$

where $v_n(t):=\widetilde{T}\cdot w_n(t)=(\widetilde{T}\circ T)\cdot u_n(t)$. Letting n tend to $+\infty$ in (3.38) and using (3.27), (3.37), (3.33) and the fact that $w_n^{(0)}\to w_0$ in H we obtain

$$\lim_{n\to+\infty}\Big\langle T\cdot\big(S\cdot\delta(\beta)\big),w_n(\beta)\Big\rangle_{H\times H}-\Big\langle(T\cdot\big(S\cdot\delta(a)\big),w_0\Big\rangle_{H\times H}-\int\limits_a^\beta\Big\langle S\cdot\Big(\frac{d\delta}{dt}(t)\Big),v(t)\Big\rangle_{X\times X^*}dt\\ +\int\limits_a^\beta\Big\langle S\cdot\big(\delta(t)\big),\Xi(t)\Big\rangle_{X\times X^*}ds=0\,,\tag{3.39}$$

for every $\delta(t)\in C^1\big([a,b];Y\big)$, where $w(t)=T\cdot u(t)$ and $v(t):=\widetilde{T}\cdot w(t)=(\widetilde{T}\circ T)\cdot u(t)$. Next since the space Y is dense in X, by approximation we deuce that for every $\bar{\delta}(t)\in C^1\big([a,b];X\big)$ such that $\bar{\delta}(b)=0$ we have

$$-\Big\langle(T\cdot\bar{\delta}(a),w_0\Big\rangle_{H\times H}-\int\limits_a^b\Big\langle\frac{d\bar{\delta}}{dt}(t),v(t)\Big\rangle_{X\times X^*}dt+\int\limits_a^b\Big\langle\bar{\delta}(t),\Xi(t)\Big\rangle_{X\times X^*}ds=0\,,\tag{3.40}$$

Thus in particular $\frac{dv}{dt}(t) \in L^{q^*}(a,b;X^*)$ and so $v(t) \in W^{1,q^*}(a,b;X^*)$. Then, as before, $w(t) \in L^\infty(a,b;H)$ and we can redefine w on a subset of $[a,b]$ of Lebesgue measure zero, so that $w(t)$ will be H-weakly continuous in t on $[a,b]$ and by (3.40) we will have $w(a) = w_0$. Moreover by (3.39) we obtain

$$\lim_{n\to+\infty} \Big\langle P\cdot y, w_n(\beta)\Big\rangle_{H\times H} = \Big\langle P\cdot y, w(\beta)\Big\rangle_{H\times H} \qquad \forall y \in Y \;\; \forall \beta \in [a,b]\,, \tag{3.41}$$

and plugging it into (3.26) we deduce

$$w_n(t) \rightharpoonup w(t) \quad \text{weakly in } H, \quad \forall t \in [a,b]\,. \tag{3.42}$$

So, by (3.40), $u(t)$ is a solution to

$$\begin{cases} \frac{dv}{dt}(t) + \Xi(t) = 0 \quad \text{for a.e. } t \in (a,b)\,, \\ w(a) = w_0\,. \end{cases} \tag{3.43}$$

Next again, since by Lemma 2.6 we have

$$\int_a^b \Big\langle u(t), \frac{dv}{dt}(t)\Big\rangle_{X\times X^*} dt = \frac{1}{2}\Big(\big\|w(b)\big\|_H^2 - \|w_0\|_H^2\Big)\,,$$

using (3.43) we obtain

$$\frac{1}{2}\big\|w(b)\big\|_H^2 + \int_a^b \Big\langle u(t), \Xi(t)\Big\rangle_{X\times X^*} dt = \frac{1}{2}\|w_0\|_H^2 \qquad \forall t \in [a,b]\,. \tag{3.44}$$

On the other hand tending n to $+\infty$ in (3.18) with $t = b$ and using (3.42) we deduce

$$\frac{1}{2}\big\|w(b)\big\|_H^2 + \varlimsup_{n\to+\infty} \int_a^b \Big\langle u_n(t), \Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*} dt \leq \frac{1}{2}\|w_0\|_H^2 \qquad \forall t \in [a,b]\,. \tag{3.45}$$

Therefore, plugging (3.44) into (3.45) we deduce

$$\varlimsup_{n\to+\infty} \int_a^b \Big\langle u_n(t), \Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*} dt \leq \int_a^b \Big\langle u(t), \Xi(t)\Big\rangle_{X\times X^*} dt\,. \tag{3.46}$$

Thus plugging (3.37) into (3.46) we obtain

$$\varlimsup_{n\to+\infty} \int_a^b \Big\langle u_n(t) - u(t), \Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*} dt \leq 0\,. \tag{3.47}$$

Next since by (3.35) we have

$$\Theta_t(x) := \Lambda_t(x) + D\Phi_t(x) \qquad \forall x \in X\,,$$

using (3.47) we deduce

$$\varlimsup_{n\to+\infty} \Bigg(\int_a^b \Big\langle u_n(t)-u(t), \Lambda_t\big(u_n(t)\big)\Big\rangle_{X\times X^*} dt + \int_a^b \Big\langle u_n(t)-u(t), D\Phi_t\big(u_n(t)\big) - \Phi_t\big(u(t)\big)\Big\rangle_{X\times X^*} dt\Bigg) \leq 0\,. \tag{3.48}$$

However, since Φ_t is convex we have

$$\int_a^b \Big\langle u_n(t) - u(t), D\Phi_t\big(u_n(t)\big) - \Phi_t\big(u(t)\big)\Big\rangle_{X\times X^*} dt \geq 0\,, \tag{3.49}$$

and plugging it into (3.48) we obtain

$$\varlimsup_{n\to+\infty} \int_a^b \Big\langle u_n(t) - u(t), \Lambda_t\big(u_n(t)\big)\Big\rangle_{X\times X^*} dt \leq 0\,. \tag{3.50}$$

I.e.
$$\varlimsup_{n\to+\infty}\Big\langle u_n(t)-u(t),\Gamma\big(u_n(t)\big)\Big\rangle_{L^q(a,b;X)\times L^{q^*}(a,b;X^*)}dt\leq 0\,. \tag{3.51}$$

where $\Gamma\big(x(t)\big)$ is defined by (3.11). On the other hand $\Gamma\big(x(t)\big)$ is weakly pseudo-monotone with respect to the evolution triple $\{X,H,X^*\}$ (see Definition 3.1). Thus by (3.42), (3.26) and (3.27), we deduce that
$$\varliminf_{n\to+\infty}\Big\langle u_n(t)-u(t),\Gamma\big(u_n(t)\big)\Big\rangle_{L^q(a,b;X)\times L^{q^*}(a,b;X^*)}dt\geq 0\,. \tag{3.52}$$

Then plugging (3.52) into (3.51) we infer
$$\lim_{n\to+\infty}\Big\langle u_n(t)-u(t),\Gamma\big(u_n(t)\big)\Big\rangle_{L^q(a,b;X)\times L^{q^*}(a,b;X^*)}dt=0\,. \tag{3.53}$$

Thus by Definition 3.1 we must have
$$\Gamma\big(u_n(t)\big)\rightharpoonup\Gamma\big(u(t)\big)\quad\text{weakly in }\;L^{q^*}(a,b;X^*)\,. \tag{3.54}$$

Moreover, by plugging (3.53) into (3.48) and using (3.49) we deduce
$$\lim_{n\to+\infty}\int_a^b\Big\langle u_n(t)-u(t),D\Phi_t\big(u_n(t)\big)\Big\rangle_{X\times X^*}dt=\\ \lim_{n\to+\infty}\int_a^b\Big\langle u_n(t)-u(t),D\Phi_t\big(u_n(t)\big)-\Phi_t\big(u(t)\big)\Big\rangle_{X\times X^*}dt=0\,. \tag{3.55}$$

On the other hand since Φ_t is convex, we have
$$\int_a^b\Phi_t\big(x(t)\big)dt\geq\int_a^b\Phi_t\big(u_n(t)\big)dt+\int_a^b\Big\langle x(t)-u_n(t),D\Phi_t\big(u_n(t)\big)\Big\rangle_{X\times X^*}dt\quad\forall\,x(t)\in L^q(a,b;X)\,, \tag{3.56}$$

Then letting $n\to+\infty$ in (3.56) and using (3.55), the convexity of Φ_t, (3.6) and (2.25) in Lemma 2.11, up to a further subsequence, we obtain
$$\int_a^b\Phi_t\big(x(t)\big)dt-\int_a^b\Phi_t\big(u(t)\big)dt-\int_a^b\Big\langle x(t)-u(t),Q(t)\Big\rangle_{X\times X^*}dt\geq 0\quad\forall\,x(t)\in L^q(a,b;X)\,, \tag{3.57}$$

where
$$D\Phi_t\big(u_n(t)\big)\rightharpoonup Q(t)\quad\text{weakly in }\;L^{q^*}(a,b;X^*)\,. \tag{3.58}$$

On the other hand $x(t):=u(t)$ is a minimizer of the l.h.s. of (3.57) and therefore by the Euler-Lagrange we must have $Q(t)\equiv D\Phi_t\big(u(t)\big)$. Plugging it into (3.58) we obtain
$$D\Phi_t\big(u_n(t)\big)\rightharpoonup D\Phi_t\big(u(t)\big)\quad\text{weakly in }\;L^{q^*}(a,b;X^*)\,. \tag{3.59}$$

Thus by (3.54), (3.59) (3.35) and (3.37) we deduce $\Xi(t)=\Theta_t\big(u(t)\big)$ for a.e. $t\in(a,b)$. Therefore, returning to (3.43) we deduce
$$\begin{cases}\frac{dv}{dt}(t)+\Theta_t\big(u(t)\big)=0\quad\text{for a.e. }t\in(a,b)\,,\\ w(a)=w_0\,.\end{cases} \tag{3.60}$$

Thus by the definition of Θ_t in (3.35) we finally deduce (3.13). □

Theorem 3.2. *Let $\{X,H,X^*\}$ be an evolution triple with the corresponding inclusion operator $T\in\mathcal{L}(X;H)$ as it was defined in Definition 2.8 together with the corresponding operator $\widetilde{T}\in\mathcal{L}(H;X^*)$ defined as in* (2.12). *Assume also that the Banach space X is separable. Furthermore, let $a,b,q\in\mathbb{R}$*

s.t. $a < b$ *and* $q \geq 2$*. Next, for every* $t \in [a,b]$ *let* $\Phi_t(x) : X \to [0,+\infty)$ *be a convex function which is Gateaux differentiable at every* $x \in X$*, satisfies* $\Phi_t(0) = 0$ *and satisfies the growth condition*

$$0 \leq \Phi_t(x) \leq C\,\|x\|_X^q + C \quad \forall x \in X,\ \forall t \in [a,b]\,, \tag{3.61}$$

for some $C > 0$*. We also assume that* $\Phi_t(x)$ *is Borel on the pair of variables* (x,t)*. Furthermore, for every* $t \in [a,b]$ *let* $\Lambda_t(x) : X \to X^*$ *be a function which is Gateaux differentiable at every* $x \in X$*,* $\Lambda_t(0) \in L^{q^*}(a,b;X^*)$ *and the derivative of* Λ_t *satisfies the growth condition*

$$\|D\Lambda_t(x)\|_{\mathcal{L}(X;X^*)} \leq g\big(\|T\cdot x\|_H\big)\left(\|x\|_X^{q-2}+1\right) \quad \forall x \in X,\ \forall t \in [a,b]\,, \tag{3.62}$$

for some nondecreasing function $g(s) : [0,+\infty) \to (0,+\infty)$*. We also assume that* $\Lambda_t(x)$ *is Borel on the pair of variables* (x,t) *(see Definition 2.2). Assume also that* Λ_t *and* Φ_t *satisfy the following monotonicity and positivity conditions*

$$\Big\langle h, \big\{D\Phi_t(x+h) - D\Phi_t(x)\big\} + D\Lambda_t(x)\cdot h\Big\rangle_{X\times X^*} \geq \\ -\,\hat{g}\big(\|T\cdot x\|_H\big)\cdot\Big(\|x\|_X^q + \mu(t)\Big)^{(2-p)/2}\cdot\|h\|_X^p\cdot\|T\cdot h\|_H^{(2-p)} \qquad \forall x \in X, \forall h \in X\ \forall t \in [a,b]\,, \tag{3.63}$$

and

$$\Big\langle x, D\Phi_t(x)+D\Lambda_t(x)\Big\rangle_{X\times X^*} \geq \frac{1}{\hat{C}}\,\|x\|_X^q - \hat{C}\Big(\|x\|_X^p+1\Big)\Big(\|T\cdot x\|_H^{(2-p)}+1\Big) - \mu(t) \quad \forall x \in X,\ \forall t \in [a,b], \tag{3.64}$$

where $p \in [0,2)$*,* $\hat{g}(s) : [0,+\infty) \to (0,+\infty)$ *is a nondecreasing function,* $\mu(t) \in L^1(a,b;\mathbb{R})$ *is a nonnegative function and* $\hat{C} > 0$ *is a constant. Finally assume that the mapping* $\Gamma\big(x(t)\big) : \big\{\bar{x}(t) \in L^q(a,b;X) :\ T\cdot\bar{x}(t) \in L^\infty(a,b;H)\big\} \to L^{q^*}(a,b;X^*)$*, defined by*

$$\Big\langle h(t), \Gamma\big(x(t)\big)\Big\rangle_{L^q(a,b;X)\times L^{q^*}(a,b;X^*)} := \int_a^b \Big\langle h(t), \Lambda_t\big(x(t)\big)\Big\rangle_{X\times X^*} dt \\ \forall\, x(t) \in \big\{\bar{x}(t) \in L^q(a,b;X) :\ T\cdot\bar{x}(t) \in L^\infty(a,b;H)\big\},\ \forall\, h(t) \in L^q(a,b;X)\,, \tag{3.65}$$

is weakly pseudo-monotone with respect to the evolution triple $\{X,H,X^*\}$ *(see Definition 3.1). Then for every* $w_0 \in H$ *there exists* $u(t) \in L^q(a,b;X)$*, such that* $w(t) := T\cdot\big(u(t)\big) \in L^\infty(a,b;H)$*,* $v(t) := \widetilde{T}\cdot\big(w(t)\big) = \widetilde{T}\circ T\big(u(t)\big) \in W^{1,q^*}(a,b;X^*)$ *and* $u(t)$ *is a solution to*

$$\begin{cases} \frac{dv}{dt}(t) + \Lambda_t\big(u(t)\big) + D\Phi_t\big(u(t)\big) = 0 \quad \textit{for a.e. } t \in (a,b)\,, \\ w(a) = w_0\,, \end{cases} \tag{3.66}$$

where we assume that $w(t)$ *is* H*-weakly continuous on* $[a,b]$*, as it was stated in Corollary 2.1. Moreover, if instead of* (3.63)*,* Λ_t *and* Φ_t *satisfy the stronger condition*

$$\Big\langle h, \big\{D\Phi_t(x+h) - D\Phi_t(x)\big\} + D\Lambda_t(x)\cdot h\Big\rangle_{X\times X^*} \geq \\ \frac{k_0\|h\|_X^2}{\hat{g}(\|T\cdot x\|_H)} - \hat{g}\big(\|T\cdot x\|_H\big)\cdot\Big(\|x\|_X^q+\mu(t)\Big)^{(2-p)/2}\cdot\|h\|_X^p\cdot\|T\cdot h\|_H^{(2-p)} \quad \forall x \in X, \forall h \in X\ \forall t \in [a,b], \tag{3.67}$$

for some constant $k_0 \geq 0$ *such that* $k_0 \neq 0$ *if* $p > 0$*, then such a solution to* (3.78) *is unique.*

Proof. Step 1: Existence of the solution. Since the Banach space is separable, by Lemma 2.10 from Appendix we deduce that there exists a separable Hilbert space Y and a bounded linear inclusion operator $S \in \mathcal{L}(Y;X)$ such that S is injective, the image of S is dense in X and moreover, S is a compact operator. Then $\{Y,X,H,X^*,Y^*\}$ is an evolution quintette with the corresponding

inclusion operators $S \in \mathcal{L}(Y,X)$, $T \in \mathcal{L}(X;H)$, $\widetilde{T} \in \mathcal{L}(H;X^*)$ and $S^* \in \mathcal{L}(X^*;Y^*)$ as it was defined in Definition 3.2 (here S^* is a dual to S operator). Next let $\Psi(y) : Y \to [0,+\infty)$ be a function defined by

$$\Psi(y) := \|y\|_Y^q + \|y\|_Y^2 \qquad \forall y \in Y. \tag{3.68}$$

Then $\Psi(y)$ is a convex function which is Gateaux differentiable on every $y \in Y$, satisfies $\Psi(0) = 0$ and satisfies the growth condition

$$(1/C_0)\,\|y\|_Y^q - C_0 \leq \Psi(y) \leq C_0\,\|y\|_Y^q + C_0 \quad \forall y \in Y\,, \tag{3.69}$$

and uniform convexity condition

$$\Big\langle y_1 - y_2, D\Psi(y_1) - D\Psi(y_2)\Big\rangle_{Y\times Y^*} \geq (1/C_0)\,\|y_1 - y_2\|_Y^2 \quad \forall y_1, y_2 \in Y, \tag{3.70}$$

for some $C_0 > 0$. Thus all the conditions of Theorem 3.1 satisfied and therefore, for every $w_0 \in H$ there exists $u(t) \in L^q(a,b;X)$, such that $w(t) := T\cdot\big(u(t)\big) \in L^\infty(a,b;H)$, $v(t) := \widetilde{T}\cdot\big(w(t)\big) = \widetilde{T}\circ T\big(u(t)\big) \in W^{1,q^*}(a,b;X^*)$, and $u(t)$ is a solution to (3.66).

Step 2: Uniqueness of the solution. Assume that Φ_t satisfies (3.67). Then applying Theorem 1.1 completes the proof. □

Remark 3.3. By Lemma 2.6 the solution to (3.66) from Theorem 3.2 satisfies the following energy equality:

$$\frac{1}{2}\|w(t)\|_H^2 + \int_a^t \Big\langle u(s), \Lambda_s\big(u(s)\big) + D\Phi_s\big(u(s)\big)\Big\rangle_{X\times X^*} ds = \frac{1}{2}\|w_0\|_H^2 \qquad \forall t \in [a,b]\,. \tag{3.71}$$

As a particular case of Theorem 3.2 we have the following Theorem.

Theorem 3.3. *Let $\{X, H, X^*\}$ be an evolution triple with the corresponding inclusion operator $T \in \mathcal{L}(X;H)$ as it was defined in Definition 2.8 together with the corresponding operator $\widetilde{T} \in \mathcal{L}(H;X^*)$ defined as in (2.12). Assume also that the Banach space X is separable. Furthermore, let $a,b,q \in \mathbb{R}$ s.t. $a < b$ and $q \geq 2$. Next, for every $t \in [a,b]$ let $\Phi_t(x) : X \to [0,+\infty)$ be a convex function which is Gateaux differentiable at every $x \in X$, satisfies $\Phi_t(0) = 0$ and satisfies the growth condition*

$$0 \leq \Phi_t(x) \leq C\,\|x\|_X^q + C \quad \forall x \in X,\ \forall t \in [a,b]\,, \tag{3.72}$$

for some $C > 0$. We also assume that $\Phi_t(x)$ is Borel on the pair of variables (x,t). Furthermore, for every $t \in [a,b]$ let $\Lambda_t(x) : X \to X^*$ be a function which is Gateaux differentiable at every $x \in X$, $\Lambda_t(0) \in L^{q^*}(a,b;X^*)$ and the derivative of Λ_t satisfies the growth condition

$$\|D\Lambda_t(x)\|_{\mathcal{L}(X;X^*)} \leq g\big(\|T\cdot x\|_H\big)\left(\|x\|_X^{q-2} + 1\right) \quad \forall x \in X,\ \forall t \in [a,b]\,, \tag{3.73}$$

for some nondecreasing function $g(s) : [0+\infty) \to (0+\infty)$. We also assume that $\Lambda_t(x)$ is Borel on the pair of variables (x,t) (see Definition 2.2). Assume also that Λ_t satisfies the following monotonicity conditions

$$\Big\langle h, D\Lambda_t(x)\cdot h\Big\rangle_{X\times X^*} \geq 0 \qquad \forall x, h \in X\ \forall t \in [a,b]\,. \tag{3.74}$$

Next let Z be a Banach space, $L \in \mathcal{L}(X,Z)$ be a compact linear operator and $G_t(z) : Z \to H$ be a function which is Gateaux differentiable on every $z \in Z$, $G_t(0) \in L^{q^*}(a,b;Z)$ and the derivative of G_t satisfies the condition

$$\|DG_t(L\cdot x)\|_{\mathcal{L}(Z;H)} \leq g\big(\|T\cdot x\|_H\big) \quad \forall x \in X,\ \forall t \in [a,b]\,, \tag{3.75}$$

for some nondecreasing function $g(s) : [0+\infty) \to (0+\infty)$. We also assume that $G_t(z)$ is strongly Borel on the pair of variables (z,t). Finally let $F_t(w) : H \to X^*$ be a function which is Gateaux differentiable on every $w \in H$, $F_t(0) \in L^{q^*}(a,b;X^*)$ and the derivative of F_t satisfies the condition

$$\|DF_t(w)\|_{\mathcal{L}(H;X^*)} \leq g\big(\|w\|_H\big) \quad \forall w \in H,\ \forall t \in [a,b]\,, \tag{3.76}$$

for some nondecreasing function $g(s) : [0+\infty) \to (0+\infty)$. We also assume that $F_t(w)$ is Borel on the pair of variables (w,t). Next assume that

$$\Big\langle x, D\Phi_t(x) + \Lambda_t(x) + \widetilde{T}\cdot G_t(L\cdot x) + F_t(T\cdot x)\Big\rangle_{X\times X^*} \geq \\ \frac{1}{\hat{C}}\|x\|_X^q - \hat{C}\big(\|x\|_X+1\big)\big(\|T\cdot x\|_H+1\big) - \mu(t) \qquad \forall x\in X,\ \forall t\in[a,b], \tag{3.77}$$

where $\hat{C}>0$ is some constant and $\mu(t)\in L^1(a,b;\mathbb{R})$ is some nonnegative function. Finally, assume that either the inclusion operator $T : X \to H$ is compact or $F_t(w)$ is weak-to-strong continuous, i.e. for every fixed $t\in[a,b]$ and every $w_n \rightharpoonup w$ weakly in H we have that $F_t(w_n)\to F_t(w)$ strongly in X^*. Then for every $w_0\in H$ there exists $u(t)\in L^q(a,b;X)$, such that $w(t) := T\cdot\big(u(t)\big)\in L^\infty(a,b;H)$, $v(t) := \widetilde{T}\cdot\big(w(t)\big) = \widetilde{T}\circ T\big(u(t)\big)\in W^{1,q^*}(a,b;X^*)$, where $q^* := q/(q-1)$, and $u(t)$ is a solution to

$$\begin{cases} \frac{dv}{dt}(t) + \Lambda_t\big(u(t)\big) + \tilde{T}\cdot G_t\big(L\cdot u(t)\big) + F_t\big(w(t)\big) + D\Phi_t\big(u(t)\big) = 0 \quad \text{for a.e. } t\in(a,b)\,, \\ w(a) = w_0\,, \end{cases} \tag{3.78}$$

where we assume that $w(t)$ is H-weakly continuous on $[a,b]$, as it was stated in Corollary 2.1. Moreover, if Φ_t satisfies the uniform convexity condition

$$\Big\langle x_1 - x_2, D\Phi_t(x_1) - D\Phi_t(x_2)\Big\rangle_{X\times X^*} \geq k_0\,\|x_1-x_2\|_X^2 \quad \forall x_1,x_2\in X, \quad \forall t\in[a,b]\,, \tag{3.79}$$

for some constant $k_0>0$, then such a solution to (3.78) is unique.

Proof. For every $t\in[a,b]$ let $\Lambda_t^{(1)}(x), \Lambda_t^{(2)}(x) : X\to X^*$ be functions defined by

$$\Lambda_t^{(1)}(x) := \tilde{T}\cdot G_t\big(L\cdot x\big)\,, \quad \Lambda_t^{(2)}(x) := F_t\big(T\cdot x\big)\,, \qquad \forall x\in X\,. \tag{3.80}$$

Then for every $j\in\{1,2\}$ for any $t\in[a,b]$ $\Lambda_t^{(j)}(x)$ is Gateaux differentiable at every $x\in X$, $\Lambda_t^{(j)}(0)\in L^\infty(a,b;X^*)$ and by (3.75) and (3.76) the derivative of $\Lambda_t^{(j)}$ satisfies the growth condition

$$\|D\Lambda_t^{(j)}(x)\|_{\mathcal{L}(X;X^*)} \leq g\big(\|T\cdot x\|_H\big)\left(\|x\|_X^{q-2}+1\right) \quad \forall x\in X,\ \forall t\in[a,b]\,. \tag{3.81}$$

Consider the mappings $\Gamma\big(x(t)\big), \Gamma^{(1)}\big(x(t)\big), \Gamma^{(2)}\big(x(t)\big) : \big\{\bar{x}(t)\in L^q(a,b;X) :\ T\cdot\bar{x}(t)\in L^\infty(a,b;H)\big\}\to L^{q^*}(a,b;X^*)$ by

$$\Big\langle h(t), \Gamma\big(x(t)\big)\Big\rangle_{L^q(a,b;X)\times L^{q^*}(a,b;X^*)} := \int_a^b \Big\langle h(t), \Lambda_t\big(x(t)\big)\Big\rangle_{X\times X^*} dt \\ \forall\, x(t)\in\big\{\bar{x}(t)\in L^q(a,b;X) :\ T\cdot\bar{x}(t)\in L^\infty(a,b;H)\big\},\ \forall\, h(t)\in L^q(a,b;X)\,, \tag{3.82}$$

and

$$\Big\langle h(t), \Gamma^{(j)}\big(x(t)\big)\Big\rangle_{L^q(a,b;X)\times L^{q^*}(a,b;X^*)} := \int_a^b \Big\langle h(t), \Lambda_t^{(j)}\big(x(t)\big)\Big\rangle_{X\times X^*} dt \\ \forall\, x(t)\in\big\{\bar{x}(t)\in L^q(a,b;X) :\ T\cdot\bar{x}(t)\in L^\infty(a,b;H)\big\},\ \forall\, h(t)\in L^q(a,b;X) \qquad \forall j\in\{1,2\}\,. \tag{3.83}$$

Then, since by (3.74), $\Gamma\big(x(t)\big)$ is monotone mapping, using (3.73) and Remark 3.1 we deduce that $\Gamma\big(x(t)\big)$ is weakly pseudo-monotone with respect to the evolution triple $\{X,H,X^*\}$ (see Definition 3.1).

Furthermore, consider the sequence $\big\{u_n(t)\big\}_{n=1}^{+\infty}\subset L^q(a,b;X)$, such that $\big\{T\cdot u_n(t)\big\}_{n=1}^{+\infty}\subset L^\infty(a,b;H)$, $u_n(t)\rightharpoonup u(t)$ weakly in $L^q(a,b;X)$, $\big\{T\cdot u_n(t)\big\}_{n=1}^{+\infty}$ is bounded in $L^\infty(a,b;H)$ and $T\cdot u_n(t)\rightharpoonup T\cdot u(t)$ weakly in H for a.e. $t\in(a,b)$. Then since the operator L is compact, by

Lemma 2.7 we deduce that $L \cdot u_n(t) \to L \cdot u(t)$ strongly in $L^q(a,b;Z)$. Thus by (3.75) we infer that $\tilde{T} \cdot G_t\big(L \cdot u_n(t)\big) \to \tilde{T} \cdot G_t\big(L \cdot u(t)\big)$ strongly in $L^{q^*}(a,b;X^*)$ and in particular

$$\lim_{n\to+\infty} \Big\langle u_n(t) - u(t), \Gamma^{(1)}\big(u_n(t)\big)\Big\rangle_{L^q(a,b;X)\times L^{q^*}(a,b;X^*)} = 0\,.$$

So, we deduce that $\Gamma^{(1)}\big(x(t)\big)$ is weakly pseudo-monotone with respect to $\{X, H, X^*\}$. Moreover, if the operator T is compact then, again by Lemma 2.7, $T \cdot u_n(t) \to T \cdot u(t)$ strongly in $L^q(a,b;H)$. Thus by (3.76) we infer in this case that $F_t\big(T \cdot u_n(t)\big) \to F_t\big(T \cdot u(t)\big)$ strongly in $L^{q^*}(a,b;X^*)$. On the other hand, if T is not compact, we have that F_t is weak to strong continuous, and the $F_t\big(T \cdot u_n(t)\big) \to F_t\big(T \cdot u(t)\big)$ strongly in X^* for a.e. fixed t. Thus, by (3.76) we also obtain in this case $F_t\big(T \cdot u_n(t)\big) \to F_t\big(T \cdot u(t)\big)$ strongly in $L^{q^*}(a,b;X^*)$. In particular, in any case,

$$\lim_{n\to+\infty} \Big\langle u_n(t) - u(t), \Gamma^{(2)}\big(u_n(t)\big)\Big\rangle_{L^q(a,b;X)\times L^{q^*}(a,b;X^*)} = 0\,.$$

Therefore, in any case, $\Gamma^{(2)}\big(x(t)\big)$ is weakly pseudo-monotone with respect to $\{X, H, X^*\}$. So, we deduce that $\Gamma\big(x(t)\big)$, $\Gamma^{(1)}\big(x(t)\big)$ and $\Gamma^{(2)}\big(x(t)\big)$ are weakly pseudo-monotone with respect to $\{X, H, X^*\}$.

Finally set

$$\bar{\Lambda}_t(x) := \Lambda_t(x) + \Lambda_t^{(1)}(x) + \Lambda_t^{(2)}(x) = \Lambda_t(x) + \tilde{T} \cdot G\big(L \cdot x\big) + F_t\big(T \cdot x\big) \quad \forall x \in X\,, \tag{3.84}$$

and define the mappings $\bar{\Gamma}\big(x(t)\big) : \big\{\bar{x}(t) \in L^q(a,b;X) :\; T \cdot \bar{x}(t) \in L^\infty(a,b;H)\big\} \to L^{q^*}(a,b;X^*)$ by

$$\Big\langle h(t), \bar{\Gamma}\big(x(t)\big)\Big\rangle_{L^q(a,b;X)\times L^{q^*}(a,b;X^*)} := \int_a^b \Big\langle h(t), \bar{\Lambda}_t\big(x(t)\big)\Big\rangle_{X\times X^*} dt$$
$$\forall\, x(t) \in \big\{\bar{x}(t) \in L^q(a,b;X) :\; T \cdot \bar{x}(t) \in L^\infty(a,b;H)\big\},\ \forall h(t) \in L^q(a,b;X)\,, \tag{3.85}$$

Then, by Remark 3.1 $\bar{\Gamma}\big(x(t)\big)$ is weakly pseudo-monotone with respect to $\{X, H, X^*\}$. Moreover, by (3.74), (3.75) and (3.76) we obtain

$$\Big\langle h, D\bar{\Lambda}_t(x) \cdot h\Big\rangle_{X\times X^*} \geq -g\big(\|T \cdot x\|_H\big) \cdot \|h\|_X \cdot \|T \cdot h\|_H \qquad \forall x \in X,\ \forall h \in X\ \forall t \in [a,b]\,, \tag{3.86}$$

Thus applying Theorem 3.2 with $\bar{\Lambda}_t$ instead of Λ_t gives the desired result. $\square$

Theorem 3.4. *Let X and Z be reflexive Banach spaces and X^* and Z^* be the corresponding dual spaces. Furthermore let H be a Hilbert space. Suppose that $Q \in \mathcal{L}(X, Z)$ is an injective inclusion operator such that its image is dense on Z. Furthermore, suppose that $P \in \mathcal{L}(Z, H)$ is an injective inclusion operator such that its image is dense on H. Let $T \in \mathcal{L}(X, H)$ be defined by $T := P \circ Q$. So $\{X, H, X^*\}$ is an evolution triple with the corresponding inclusion operator $T \in \mathcal{L}(X;H)$ as it was defined in Definition 2.8 together with the corresponding operator $\widetilde{T} \in \mathcal{L}(H;X^*)$ defined as in (2.12). Assume also that the Banach space X is separable. Next let $a, b \in \mathbb{R}$ s.t. $a < b$ and $q \geq 2$. Furthermore, for every $t \in [a,b]$ let $\Lambda_t(z) : Z \to X^*$ and $A_t(z) : Z \to X^*$ be functions which are Gateaux differentiable at every $z \in Z$ and $\Lambda_t(0), A_t(0) \in L^{q^*}\big(a,b;X^*\big)$. Assume that for every $t \in [a,b]$, they satisfy the following bounds*

$$\big\|D\Lambda_t(z)\big\|_{\mathcal{L}(Z;X^*)} \leq g\big(\|P \cdot z\|_H\big) \cdot \Big(\|z\|_Z^{q-2} + 1\Big) \qquad \forall z \in Z,\ \forall t \in [a,b]\,, \tag{3.87}$$

$$\big\|\Lambda_t(z)\big\|_{X^*} \leq g\big(\|P \cdot z\|_H\big) \cdot \Big(\|L_0 \cdot z\|_{V_0}^{q-1} + 1\Big) \qquad \forall z \in Z,\ \forall t \in [a,b]\,, \tag{3.88}$$

and

$$\big\|DA_t(z)\big\|_{\mathcal{L}(Z;X^*)} \leq g\big(\|P \cdot z\|_H\big) \cdot \Big(\|L_0 \cdot z\|_{V_0}^{q-2} + 1\Big) \qquad \forall z \in Z,\ \forall t \in [a,b]\,, \tag{3.89}$$

where $g(s):[0,+\infty)\to(0,+\infty)$ is some nondecreasing function, V_0 is some Banach space and $L_0\in\mathcal{L}(Z;V_0)$ is some compact linear operator. Moreover, assume that Λ_t and A_t satisfy the following monotonicity and positivity conditions

$$\Big\langle h, DA_t(z)\cdot(Q\cdot h)+D\Lambda_t(z)\cdot(Q\cdot h)\Big\rangle_{X\times X^*}\geq -\hat{g}\big(\|P\cdot z\|_H\big)\cdot\Big(\|z\|_Z^q+\mu(t)\Big)^{(2-p)/2}\cdot\big\|h\big\|_X^p\cdot\big\|T\cdot h\big\|_H^{(2-p)}$$
$$\forall z\in Z,\ \forall h\in X,\ \forall t\in[a,b],\quad (3.90)$$

and

$$\Big\langle h, A_t(Q\cdot h)+\Lambda_t\big(Q\cdot h\big)\Big\rangle_{X\times X^*}\geq(1/\bar{C})\big\|Q\cdot h\big\|_Z^q-\bar{C}\Big(\big\|Q\cdot h\big\|_Z^p+1\Big)\cdot\Big(\big\|T\cdot h\big\|_H^{(2-p)}+1\Big)-\mu(t)$$
$$\forall h\in X\,,\ \forall t\in[a,b],\quad (3.91)$$

where $p\in[0,2)$, $\hat{g}(s):[0,+\infty)\to(0,+\infty)$ is some nondecreasing function, $\mu(t)\in L^1(a,b;\mathbb{R})$ is some nonnegative function and $\bar{C}>0$ is some constant. We also assume that $\Lambda_t(z)$ $A_t(z)$ are Borel on the pair of variables (z,t). Finally assume that there exists a family of Banach spaces $\{V_j\}_{j=1}^{+\infty}$ and a family of compact bounded linear operators $\{L_j\}_{j=1}^{+\infty}$, where $L_j\in\mathcal{L}(Z,V_j)$, which satisfy the following condition:

- *If $\{h_n\}_{n=1}^{+\infty}\subset Z$ is a sequence and $h_0\in Z$, are such that for every fixed j $\lim_{n\to+\infty}L_j\cdot h_n=L_j\cdot h_0$ strongly in V_j and $P\cdot h_n\rightharpoonup P\cdot h_0$ weakly in H, then for every fixed $t\in(a,b)$ we have $\Lambda_t(h_n)\rightharpoonup\Lambda_t(h_0)$ weakly in X^* and $DA_t(h_n)\to DA_t(h_0)$ strongly in $\mathcal{L}(Z,X^*)$.*

Then for every $w_0\in H$ there exists $z(t)\in L^q(a,b;Z)$ such that $w(t):=P\cdot z(t)\in L^\infty(a,b;H)$, $v(t):=\widetilde{T}\cdot\big(w(t)\big)\in W^{1,q^*}(a,b;X^*)$ and $z(t)$ satisfies the following equation

$$\begin{cases}\frac{dv}{dt}(t)+A_t\big(z(t)\big)+\Lambda_t\big(z(t)\big)=0 \quad \text{for a.e. } t\in(a,b)\,,\\ w(a)=w_0\,,\end{cases}\tag{3.92}$$

where we assume that $w(t)$ is H-weakly continuous on $[a,b]$, as it was stated in Corollary 2.1. Moreover, if in addition we assume that there exist a Banach space V, a compact operator $L\in\mathcal{L}(Z,V)$, a nondecreasing function $\widetilde{g}(s):[0,+\infty)\to(0,+\infty)$ and for every $t\in[a,b]$ a convex Gateux differentiable functions $\Phi_t:Z\to\mathbb{R}$, Borel measurable on (z,t), and a Gateux differentiable mapping $F_t(\sigma):V\to Z^*$, Borel measurable on (σ,t), satisfying $F_t(0)\in L^{q^*}(a,b;Z^*)$ and such that

$$0\leq\Phi_t(z)\leq\widetilde{g}\big(\|P\cdot z\|_H\big)\cdot\big(\|z\|_Z^q+1\big)\quad\forall z\in Z,\ \forall t\in[a,b]\,,\tag{3.93}$$

$$\big\|DF_t(L\cdot z)\big\|_{\mathcal{L}(V;Z^*)}\leq\widetilde{g}\big(\|P\cdot z\|_H\big)\cdot\big(\|L\cdot z\|_V^{q-2}+1\big)\quad\forall z\in Z,\ \forall t\in[a,b]\,,\tag{3.94}$$

and

$$\Big\langle h, A_t(Q\cdot h)+\Lambda_t\big(Q\cdot h\big)\Big\rangle_{X\times X^*}\geq\Phi_t\big(Q\cdot h\big)+\Big\langle Q\cdot h, F_t\big((L\circ Q)\cdot h\big)\Big\rangle_{Z\times Z^*}\qquad\forall h\in X\,,\ \forall t\in[a,b]\,,\tag{3.95}$$

then the function $z(t)$, as above, satisfies the following energy inequality

$$\frac{1}{2}\big\|w(t)\big\|_H^2+\int_a^t\bigg(\Phi_s\big(z(s)\big)+\Big\langle z(s),F_s\big(L\cdot z(s)\big)\Big\rangle_{Z\times Z^*}\bigg)ds\leq\frac{1}{2}\big\|w_0\big\|_H^2\qquad\forall t\in[a,b]\,.\tag{3.96}$$

Proof. As before, since the Banach space is separable, by Lemma 2.10 from Appendix we deduce that there exists a separable Hilbert space Y and a bounded linear inclusion operator $S\in\mathcal{L}(Y;X)$ such that S is injective (i.e. $\ker S=\{0\}$), the image of S is dense in X and moreover, S is a compact operator. Then $\{Y,X,H,X^*,Y^*\}$ is an evolution quintette with the corresponding inclusion operators $S\in\mathcal{L}(Y,X)$, $T\in\mathcal{L}(X;H)$, $\widetilde{T}\in\mathcal{L}(H;X^*)$ and $S^*\in\mathcal{L}(X^*;Y^*)$ as it was defined in Definition 3.2 (here S^* is a dual to S operator). Next let $\Psi(y):Y\to[0,+\infty)$ be a function defined by

$$\Psi(y):=\|y\|_Y^q+\|y\|_Y^2\qquad\forall y\in Y.\tag{3.97}$$

Then $\Psi(y)$ is a convex function which is Gateaux differentiable on every $y \in Y$, satisfies $\Psi(0) = 0$ and satisfies the uniform convexity condition. Then by applying Lemma 3.1 we deduce that for every $\psi_0 \in Y$ and every $\varepsilon > 0$ there exists $\psi(t) \in \mathcal{R}_{Y,q}(a,b)$ such that $w(a) = (T \circ S) \cdot \psi_0$ and $\psi(t)$ is a solution of

$$\frac{d\varphi}{dt}(t) + S^* \cdot \Big(A_t\big(z(t)\big) + \Lambda_t\big(z(t)\big)\Big) + \varepsilon D\Psi\big(\psi(t)\big) = 0 \quad \text{for a.e. } t \in (a,b)\,, \tag{3.98}$$

where $u(t) := S \cdot \big(\psi(t)\big)$, $z(t) := (Q \circ S) \cdot \big(\psi(t)\big) = Q \cdot \big(u(t)\big)$, $w(t) := (T \circ S) \cdot \big(\psi(t)\big) = P \cdot \big(z(t)\big)$, $\varphi(t) := (S^* \circ \widetilde{T} \circ T \circ S) \cdot \big(\psi(t)\big) = (S^* \circ \widetilde{T}) \cdot \big(w(t)\big)$ and we assume that $w(t)$ is H-weakly continuous on $[a,b]$, as it was stated in Corollary 2.1.

Next let $w_0 \in H$. Then, since the image of the operator $T \circ S$ is dense in H, there exists a sequence $\{\psi_n^{(0)}\} \subset Y$ such that $w_n^{(0)} := (T \circ S) \cdot \psi_n^{(0)} \to w_0$ strongly in H as $n \to +\infty$. Furthermore, let $\varepsilon_n \to 0^+$ as $n \to +\infty$. Set $\psi_n(t) \in \mathcal{R}_{Y,q}(a,b)$ be a solution of

$$\begin{cases} \frac{d\varphi_n}{dt}(t) + S^* \cdot \Big(A_t\big(z_n(t)\big) + \Lambda_t\big(z_n(t)\big)\Big) + \varepsilon_n D\Psi\big(\psi_n(t)\big) = 0 \;\text{ for } t \in (a,b)\,, \\ w_n(a) = (T \circ S) \cdot \psi_n^{(0)}\,, \end{cases} \tag{3.99}$$

where $u_n(t) := S \cdot \big(\psi_n(t)\big)$, $z_n(t) := (Q \circ S) \cdot \big(\psi_n(t)\big) = Q \cdot \big(u_n(t)\big)$, $w_n(t) := (T \circ S) \cdot \big(\psi_n(t)\big) = P \cdot \big(z_n(t)\big)$, $\varphi_n(t) := (S^* \circ \widetilde{T} \circ T \circ S) \cdot \big(\psi_n(t)\big) = (S^* \circ \widetilde{T}) \cdot \big(w_n(t)\big)$ and we assume that $w_n(t)$ is H-weakly continuous on $[a,b]$ (As we saw above such a solution exists). Then by (3.99) we deduce

$$\int_a^t \Big\langle \psi_n(s), \frac{d\varphi_n}{dt}(s) \Big\rangle_{Y \times Y^*} ds + \int_a^t \Big\langle u_n(s), A_s\big(z_n(s)\big) + \Lambda_s\big(z_n(s)\big) \Big\rangle_{X \times X^*} ds \\ + \varepsilon_n \int_a^t \Big\langle \psi_n(s), D\Psi\big(\psi_n(s)\big) \Big\rangle_{Y \times Y^*} ds = 0 \qquad \forall t \in [a,b]\,. \tag{3.100}$$

However, since by Lemma 2.6 we have

$$\int_a^t \Big\langle \psi_n(s), \frac{d\varphi_n}{dt}(s) \Big\rangle_{Y \times Y^*} ds = \frac{1}{2}\Big(\big\|w_n(t)\big\|_H^2 - \big\|w_n^{(0)}\big\|_H^2\Big)\,,$$

using (3.100) we obtain

$$\frac{1}{2}\big\|w_n(t)\big\|_H^2 + \int_a^t \Big\langle u_n(s), A_s\big(z_n(s)\big) + \Lambda_s\big(z_n(s)\big) \Big\rangle_{X \times X^*} ds \\ + \varepsilon_n \int_a^t \Big\langle \psi_n(s), D\Psi\big(\psi_n(s)\big) \Big\rangle_{Y \times Y^*} ds = \frac{1}{2}\big\|w_n^{(0)}\big\|_H^2 \qquad \forall t \in [a,b]\,. \tag{3.101}$$

However, as before, since $\Psi(\cdot)$ is convex and since $\Psi(\cdot) \geq 0$, $\Psi(0) = 0$ and then also $D\Psi(0) = 0$ we have

$$\Big\langle \psi_n(t), D\Psi\big(\psi_n(t)\big) \Big\rangle_{Y \times Y^*} \geq \Psi\big(\psi_n(t)\big) \geq 0 \qquad \forall t \in (a,b)\,. \tag{3.102}$$

Therefore, using (3.102), from (3.101) we deduce

$$\frac{1}{2}\big\|w_n(t)\big\|_H^2 + \int_a^t \Big\langle u_n(s), A_s\big(z_n(s)\big) + \Lambda_s\big(z_n(s)\big) \Big\rangle_{X \times X^*} ds \leq \frac{1}{2}\big\|w_n^{(0)}\big\|_H^2 \qquad \forall t \in [a,b]\,. \tag{3.103}$$

Thus, inserting (3.91) into (3.103) we deduce that there exists $C_1 > 0$ such that

$$\big\|w_n(t)\big\|_H^2 + \int_a^t \big\|z_n(s)\big\|_Z^q ds \leq C_1 + C_1\bigg(\int_a^t \big\|w_n(s)\big\|_H^2 ds\bigg)^{(2-p)/2} \cdot \bigg(\int_a^t \big\|z_n(s)\big\|_Z^2 ds\bigg)^{p/2} \qquad \forall t \in [a,b]\,. \tag{3.104}$$

In particular,

$$\int_a^t \big\|z_n(s)\big\|_Z^2 ds \leq \bar{C}_1 + \bar{C}_1\bigg(\int_a^t \big\|w_n(s)\big\|_H^2 ds\bigg)^{(2-p)/2} \cdot \bigg(\int_a^t \big\|z_n(s)\big\|_Z^2 ds\bigg)^{p/2} \qquad \forall t \in [a,b]\,. \tag{3.105}$$

Therefore, by (3.105), there exists a constant $C_2 > 0$ such that

$$\int_a^t \big\|z_n(s)\big\|_Z^2 ds \leq C_2 + C_2 \int_a^t \big\|w_n(s)\big\|_H^2 ds \qquad \forall t \in [a,b]\,. \tag{3.106}$$

Plugging (3.106) into (3.104) we deduce

$$\big\|w_n(t)\big\|_H^2 \leq C_3 + C_3 \int_a^t \big\|w_n(s)\big\|_H^2 ds \qquad \forall t \in [a,b]\,. \tag{3.107}$$

Thus, as before in (3.23), (3.24) and (3.25) from (3.107) we obtain that sequence $\{w_n(t)\}$ is bounded in $L^\infty(a,b;H)$. Then by (3.106) we deduce that that sequence $\{z_n(t)\}$ is bounded in $L^2(a,b;Z)$ and then by (3.104) we also deduce that that sequence $\{z_n(t)\}$ is bounded in $L^q(a,b;Z)$. Therefore in particular, up to a subsequence we have

$$\begin{cases} z_n(t) \rightharpoonup z(t) \quad \text{weakly in } \ L^q(a,b;Z)\,, \\ w_n(t) \rightharpoonup w(t) \quad \text{weakly in } \ L^q(a,b;H)\,, \\ v_n(t) \rightharpoonup v(t) \quad \text{weakly in } \ L^q(a,b;X^*)\,, \end{cases} \tag{3.108}$$

where $v_n(t) := \widetilde{T} \cdot w_n(t) = (\widetilde{T} \circ T) \cdot u_n(t)$ and $w(t) := P \cdot z(t)$, $v(t) := \widetilde{T} \cdot w(t)$. Next plugging (3.108) into (3.101) and using (3.91) and the facts that $\{w_n(t)\}$ is bounded in $L^\infty(a,b;H)$ and $\{z_n(t)\}$ is bounded in $L^q(a,b;Z)$, we deduce

$$\varepsilon_n \int_a^b \Big\langle \psi_n(s), D\Psi\big(\psi_n(s)\big)\Big\rangle_{Y \times Y^*} ds \leq C_4\,, \tag{3.109}$$

where C_4 is a constant. Then using (3.102) and (3.97), we deduce from (3.109),

$$\varepsilon_n \int_a^b \big\|\psi_n(s)\big\|_Y^q ds \leq C_5\,. \tag{3.110}$$

Next by (2.25) in Lemma 2.11, for some $\bar{C} > 0$ we have

$$\Big\|D\Psi\big(\psi_n(t)\big)\Big\|_{Y^*} \leq \bar{C}\big\|\psi_n(t)\big\|_Y^{q-1} + \bar{C} \qquad \forall t \in (a,b)\,, \tag{3.111}$$

and then

$$\Big\|D\Psi\big(\psi_n(t)\big)\Big\|_{Y^*}^{q^*} \leq \bar{C}_0\big\|\psi_n(t)\big\|_Y^{q} + \bar{C}_0 \qquad \forall t \in (a,b)\,, \tag{3.112}$$

Thus plugging (3.112) into (3.110) we deduce

$$\int_a^b \Big\|\varepsilon_n D\Psi\big(\psi_n(s)\big)\Big\|_{Y^*}^{q^*} ds \leq \hat{C}\varepsilon_n^{1/(q-1)}\,. \tag{3.113}$$

So,

$$\lim_{n\to+\infty} \Big\|\varepsilon_n D\Psi\big(\psi_n(t)\big)\Big\|_{L^{q^*}(a,b;Y^*)} = 0\,. \tag{3.114}$$

In particular, plugging (3.108), (3.89), (3.114), (3.88) and the fact that $\{w_n(t)\}$ is bounded in $L^\infty(a,b;H)$ into (3.99) we deduce that

$$\frac{d\varphi_n}{dt}(t) \quad \text{is bounded in } \ L^{q^*}(a,b;Y^*)\,. \tag{3.115}$$

Next by (3.88), (3.89) and (3.108), up to a further subsequence, we must have

$$\begin{cases} A_t\big(z_n(t)\big) \rightharpoonup \bar{A}(t) \quad \text{weakly in } \ L^q(a,b;X^*)\,, \\ \Lambda_t\big(z_n(t)\big) \rightharpoonup \bar{\Lambda}(t) \quad \text{weakly in } \ L^{q^*}(a,b;X^*)\,. \end{cases} \tag{3.116}$$

On the other hand by (3.99) and by Corollary 2.1 for any $\beta \in [a,b]$ and every $\delta(t) \in C^1\big([a,b];Y\big)$ we have

$$\Big\langle T\cdot\big(S\cdot\delta(\beta)\big), w_n(\beta)\Big\rangle_{H\times H} - \Big\langle T\cdot\big(S\cdot\delta(a)\big), w_n^{(0)}\Big\rangle_{H\times H} - \int\limits_a^\beta \Big\langle S\cdot\Big(\frac{d\delta}{dt}(t)\Big), v_n(t)\Big\rangle_{X\times X^*} dt+ \\ \int\limits_a^\beta \Big\langle \delta(t), \varepsilon_n D\Psi\big(\psi_n(t)\big)\Big\rangle_{Y\times Y^*} dt + \int\limits_a^\beta \Big\langle S\cdot\delta(t), A_t\big(z_n(t)\big)+\Lambda_t\big(z_n(t)\big)\Big\rangle_{X\times X^*} ds = 0\,, \quad (3.117)$$

Letting n tend to $+\infty$ in (3.117) and using (3.108), (3.116), (3.114) and the fact that $w_n^{(0)} \to w_0$ in H we obtain

$$\lim_{n\to+\infty} \Big\langle T\cdot\big(S\cdot\delta(\beta)\big), w_n(\beta)\Big\rangle_{H\times H} - \Big\langle (T\cdot\big(S\cdot\delta(a)\big), w_0\Big\rangle_{H\times H} - \int\limits_a^\beta \Big\langle S\cdot\Big(\frac{d\delta}{dt}(t)\Big), v(t)\Big\rangle_{X\times X^*} dt \\ + \int\limits_a^\beta \Big\langle S\cdot\big(\delta(t)\big), \bar A(t)+\bar\Lambda(t)\Big\rangle_{X\times X^*} ds = 0\,, \quad (3.118)$$

for every $\delta(t) \in C^1\big([a,b];Y\big)$. Next since the space Y is dense in X, by approximation we deuce that for every $\bar\delta(t) \in C^1\big([a,b];X\big)$ such that $\bar\delta(b)=0$ we have

$$-\Big\langle (T\cdot\bar\delta(a), w_0\Big\rangle_{H\times H} - \int\limits_a^b \Big\langle \frac{d\bar\delta}{dt}(t), v(t)\Big\rangle_{X\times X^*} dt + \int\limits_a^b \Big\langle \bar\delta(t), \bar A(t)+\bar\Lambda(t)\Big\rangle_{X\times X^*} ds = 0\,. \quad (3.119)$$

Thus in particular $\frac{dv}{dt}(t) \in L^{q^*}(a,b;X^*)$ and so $v(t) \in W^{1,q^*}(a,b;X^*)$. Then, since $\{w_n(t)\}$ is bounded in $L^\infty(a,b;H)$, we have $w(t) \in L^\infty(a,b;H)$ and thus, as before, we can redefine w on a subset of $[a,b]$ of Lebesgue measure zero, so that $w(t)$ will be H-weakly continuous in t on $[a,b]$ and by (3.119) we will have $w(a)=w_0$. So $w(t)$ is a solution to the following equation

$$\begin{cases} \frac{dv}{dt}(t) + \bar A(t) + \bar\Lambda(t) = 0 \quad \text{for a.e. } t\in(a,b)\,. \\ w(a) = w_0\,, \end{cases} \quad (3.120)$$

Thus in particular for any $\beta \in [a,b]$ and every $\delta(t) \in C^1\big([a,b];Y\big)$ we have

$$\Big\langle T\cdot\big(S\cdot\delta(\beta)\big), w(\beta)\Big\rangle_{H\times H} - \Big\langle (T\cdot\big(S\cdot\delta(a)\big), w_0\Big\rangle_{H\times H} - \int\limits_a^\beta \Big\langle S\cdot\Big(\frac{d\delta}{dt}(t)\Big), v(t)\Big\rangle_{X\times X^*} dt \\ + \int\limits_a^\beta \Big\langle S\cdot\big(\delta(t)\big), \bar A(t)+\bar\Lambda(t)\Big\rangle_{X\times X^*} ds = 0\,. \quad (3.121)$$

Plugging (3.121) into (3.118) we deduce

$$\lim_{n\to+\infty} \Big\langle (T\circ S)\cdot y, w_n(\beta)\Big\rangle_{H\times H} = \Big\langle (T\circ S)\cdot y, w(\beta)\Big\rangle_{H\times H} \quad \forall y\in Y\ \forall \beta\in[a,b]\,. \quad (3.122)$$

Therefore, since the image of $(T\circ S)$ has dense range in H and $\{w_n(t)\}$ is bounded in $L^\infty(a,b;H)$ we deduce that

$$w_n(t) \rightharpoonup w(t) \quad \text{weakly in } H \qquad \forall t\in[a,b]\,. \quad (3.123)$$

Next there exists a family of reflexive Banach spaces $\{V_j\}_{j=1}^{+\infty}$ and a family of compact bounded linear operators $\{L_j\}_{j=1}^{+\infty}$, where $L_j \in \mathcal{L}(Z,V_j)$, which satisfy the following condition:

- If $\{h_n\}_{n=1}^{+\infty} \subset Z$ is a sequence and $h_0 \in Z$, are such that for every fixed j $\lim_{n\to+\infty} L_j \cdot h_n = L_j \cdot h_0$ strongly in V_j and $P \cdot h_n \rightharpoonup P \cdot h_0$ weakly in H, then for every fixed $t \in (a,b)$ $\Lambda_t(h_n) \rightharpoonup \Lambda_t(h_0)$ weakly in X^* and $DA_t(h_n) \to DA_t(h_0)$ strongly in $\mathcal{L}(Z, X^*)$.

Therefore, using (3.108), (3.115) and Lemma 2.9, we deduce that for every j we have $L_j \cdot z_n(t) \to L_j \cdot z(t)$ strongly in $L^q(a,b;V_j)$ as $n \to +\infty$. By the same way we obtain $L_0 \cdot z_n(t) \to L_0 \cdot z(t)$ strongly in $L^q(a,b;V_0)$ as $n \to +\infty$. Thus, up to a further subsequence we will have $L_j \cdot z_n(t) \to L_j \cdot z(t)$ strongly in V_j for almost every $t \in (a,b)$ and every j. Therefore, by (3.123) and the above condition, we must have $\Lambda_t\big(z_n(t)\big) \rightharpoonup \Lambda_t\big(z(t)\big)$ weakly in X^* and $DA_t\big(sz_n(t) + (1-s)z(t)\big) \to DA_t\big(z(t)\big)$ strongly in $\mathcal{L}(Z, X^*)$ for almost every $t \in (a,b)$ and for every $s \in [0,1]$. Therefore, using (3.88), the fact that $\{w_n(t)\}$ is bounded in $L^\infty(a,b;H)$ and the fact that $L_0 \cdot z_n(t) \to L_0 \cdot z(t)$ strongly in $L^q(a,b;V_0)$, we deduce that

$$\int_a^b \Big\langle h(t), \Lambda_t\big(z_n(t)\big)\Big\rangle_{X\times X^*} dt \to \int_a^b \Big\langle h(t), \Lambda_t\big(z(t)\big)\Big\rangle_{X\times X^*} dt \qquad \forall h(t) \in L^q(a,b,X)\,.$$

Thus

$$\Lambda_t\big(z_n(t)\big) \rightharpoonup \Lambda_t\big(z(t)\big) \quad \text{weakly in } \; L^{q^*}(a,b;X^*)\,. \tag{3.124}$$

In the similar way, by (3.89), the fact that $\{w_n(t)\}$ is bounded in $L^\infty(a,b;H)$ and the fact that $L_0 \cdot z_n(t) \to L_0 \cdot z(t)$ strongly in $L^q(a,b;V_0)$, we deduce that, for $q = 2$ we have

$$\begin{aligned} DA_t\big(sz_n(t) + (1-s)z(t)\big) \to DA_t\big(z(t)\big) \quad \text{strongly in } \mathcal{L}(Z,X^*) \text{ for a.e. } t \in (a,b)\ \forall s \in [0,1] \\ \text{and} \quad DA_t\big(sz_n(t) + (1-s)z(t)\big) \text{ is bounded in } L^\infty\big(a,b;\mathcal{L}(Z,X^*)\big) \text{ uniformly by } s\,. \end{aligned} \tag{3.125}$$

and for $q > 2$ we have

$$DA_t\big(sz_n(t) + (1-s)z(t)\big) \to DA_t\big(z(t)\big) \;\text{ strongly in }\; L^{q/(q-2)}\big(a,b;\mathcal{L}(Z,X^*)\big) \;\; \forall s \in [0,1]\,. \tag{3.126}$$

In both cases

$$\begin{aligned} \Big\{DA_t\big(sz_n(t) + (1-s)z(t)\big)\Big\}^* \cdot h(t) \to \Big\{DA_t\big(z(t)\big)\Big\}^* \cdot h(t) \;\text{ strongly in }\; L^{q^*}(a,b,Z) \\ \forall h(t) \in L^q(a,b,X) \;\; \forall s \in [0,1]\,, \end{aligned} \tag{3.127}$$

where $\big\{DA_t(\cdot)\big\}^* \in \mathcal{L}(X,Z^*)$ is the adjoint to $DA_t(\cdot) \in \mathcal{L}(Z,X^*)$ operator. Thus, by (3.89), the fact that $\{w_n(t)\}$ is bounded in $L^\infty(a,b;H)$ and the fact that $L_0 \cdot z_n(t) \to L_0 \cdot z(t)$ strongly in $L^q(a,b;V_0)$, together with (3.127) and (3.108) we obtain

$$\begin{aligned} &\int_a^b \Big\langle h(t), A_t\big(z_n(t)\big) - A_t\big(z(t)\big)\Big\rangle_{X\times X^*} dt = \\ &\qquad \int_0^1\int_a^b \Big\langle h(t), DA_t\Big(sz_n(t) + (1-s)z(t)\Big)\cdot\Big(z_n(t) - z(t)\Big)\Big\rangle_{X\times X^*} dtds = \\ &\int_0^1\int_a^b \Big\langle \Big(z_n(t) - z(t)\Big), \Big\{DA_t\big(sz_n(t) + (1-s)z(t)\big)\Big\}^* \cdot h(t)\Big\rangle_{Z\times Z^*} dtds \to 0 \qquad \forall h(t) \in L^q(a,b,X)\,. \end{aligned}$$

So, by (3.116), and (3.124) we have $\bar{\Lambda}(t) = \Lambda_t\big(z(t)\big)$ and $\bar{A}(t) = A_t\big(z(t)\big)$, and thus using (3.120) we finally deduce that $z(t)$ is a solution to (3.92).

Finally, assume that there exist a reflexive Banach space V, a compact operator $L \in \mathcal{L}(Z,V)$, and for every $t \in [a,b]$ a convex Gateux differentiable functions $\Phi_t : Z \to \mathbb{R}$ and a Gateux differentiable mapping $F_t(\sigma) : V \to Z^*$ satisfying (3.93), (3.94) and (3.95). Then, since, as before, we have $L \cdot z_n(t) \to L \cdot z(t)$ strongly in $L^q(a,b;V)$, we deduce that, up to a subsequence, $F_t\big(L \cdot z_n(t)\big) \to F_t\big(L \cdot z(t)\big)$ strongly in $L^{q^*}(a,b;Z^*)$. On the other hand by (3.95) and (3.103) we infer

$$\frac{1}{2}\big\|w_n(t)\big\|_H^2 + \int_a^t \left(\Phi_s\big(z_n(s)\big) + \Big\langle z_n(s), F_s\big(L\cdot z_n(s)\big)\Big\rangle_{Z\times Z^*}\right) ds \le \frac{1}{2}\big\|w_n^{(0)}\big\|_H^2 \qquad \forall t \in [a,b]\,. \tag{3.128}$$

Therefore, letting n tend to $+\infty$ in (3.128) and using (3.123), (3.108) and the convexity of Φ_t we finally obtain (3.96). □

As a particular case of Theorem 3.4 we have the following Theorem.

Theorem 3.5. *Let X and Z be reflexive Banach spaces and X^* and Z^* be the corresponding dual spaces. Furthermore let H be a Hilbert space. Suppose that $Q \in \mathcal{L}(X,Z)$ is an injective inclusion operator such that its image is dense on Z. Furthermore, suppose that $P \in \mathcal{L}(Z,H)$ is an injective inclusion operator such that its image is dense on H. Let $T \in \mathcal{L}(X,H)$ be defined by $T := P \circ Q$. So $\{X, H, X^*\}$ is an evolution triple with the corresponding inclusion operator $T \in \mathcal{L}(X;H)$ as it was defined in Definition 2.8 together with the corresponding operator $\widetilde{T} \in \mathcal{L}(H;X^*)$ defined as in (2.12). Assume also that the Banach space X is separable. Next let $a,b \in \mathbb{R}$ s.t. $a < b$. Furthermore, for every $t \in [a,b]$ let $\Lambda_t \in L^\infty\big(a,b;\mathcal{L}(Z,X^*)\big)$. which satisfies the following positivity condition*

$$\Big\langle h, \Lambda_t \cdot (Q \cdot h)\Big\rangle_{X \times X^*} \geq 0 \qquad \forall h \in X\,,\ \forall t \in [a,b]\,. \tag{3.129}$$

Next let V be a Banach space, $L \in \mathcal{L}(Z,V)$ be a compact linear operator and $G_t(h) : V \to H$ be a function which is Gateaux differentiable on every $h \in V$, $G_t(0) \in L^2(a,b;V)$ and the derivative of G_t satisfies the condition

$$\|DG_t(L \cdot h)\|_{\mathcal{L}(V;H)} \leq g\big(\|P \cdot h\|_H\big) \quad \forall h \in Z,\ \forall t \in [a,b]\,, \tag{3.130}$$

for some nondecreasing function $g(s) : [0,+\infty) \to (0,+\infty)$. We also assume that $G_t(h)$ is Borel on the pair of variables (h,t) (see Definition 2.2). Next let $F_t(w) : H \to X^*$ be a function which is Gateaux differentiable at every $w \in H$ for every $t \in [a,b]$, and satisfies $F_t(0) \in L^2(a,b;X^*)$ and the Lipshitz condition

$$\big\|DF_t(w)\big\|_{\mathcal{L}(H;X^*)} \leq g\big(\|w\|_H\big) \qquad \forall w \in H,\ \forall t \in [a,b]\,, \tag{3.131}$$

for some nondecreasing function $g(s) : [0,+\infty) \to (0,+\infty)$. We also assume that $F_t(w)$ is Borel on the pair of variables (w,t). Moreover, suppose that Λ_t G_t and F_t satisfy the following lower bound condition

$$\begin{aligned} &\Big\langle h, \Lambda_t \cdot (Q \cdot h) + \widetilde{T} \cdot G_t\big((L \circ Q) \cdot h\big) + F_t\big(T \cdot h\big)\Big\rangle_{X \times X^*} \geq \\ &\quad (1/\bar{C})\big\|Q \cdot h\big\|_Z^2 - \bar{C}\Big(\big\|Q \cdot h\big\|_Z^p + 1\Big) \cdot \Big(\big\|T \cdot h\big\|_H^{(2-p)} + 1\Big) - \mu(t) \quad \forall h \in X\,,\ \forall t \in [a,b], \end{aligned} \tag{3.132}$$

for some constants $p \in [0,2)$ and $\bar{C} > 0$ and a nonnegative function $\mu(t) \in L^1(a,b;\mathbb{R})$. Finally assume that there exists a family of reflexive Banach spaces $\{V_j\}_{j=1}^{+\infty}$ and a family of bounded linear operators $\{L_j\}_{j=1}^{+\infty}$, where $L_j \in \mathcal{L}(H,V_j)$, which satisfy the following two conditions:

(a) For all j the operator $L_j \circ P$ is compact.

(b) If $\{h_n\}_{n=1}^{+\infty} \subset H$ is a sequence such that for every fixed j $\lim_{n\to+\infty} L_j \cdot h_n = L_j \cdot h_0$ strongly in V_j and $h_n \rightharpoonup h_0$ weakly in H, then for every fixed $t \in (a,b)$ $F_t(h_n) \rightharpoonup F_t(h_0)$ weakly in X^*.

Then for every $w_0 \in H$ there exists $z(t) \in L^2(a,b;Z)$ such that $w(t) := P \cdot z(t) \in L^\infty(a,b;H)$, $v(t) := \widetilde{T} \cdot \big(w(t)\big) \in W^{1,2}(a,b;X^*)$ and $z(t)$ satisfies the following equation

$$\begin{cases} \frac{dv}{dt}(t) + \Lambda_t \cdot \big(z(t)\big) + \widetilde{T} \cdot G_t\big(L \cdot z(t)\big) + F_t\big(w(t)\big) = 0 & \text{for a.e. } t \in (a,b)\,, \\ w(a) = w_0\,, \end{cases} \tag{3.133}$$

where we assume that $w(t)$ is H-weakly continuous on $[a,b]$, as it was stated in Corollary 2.1. Moreover if we assume in addition that there exist a reflexive Banach space E, a compact operator $L_0 \in \mathcal{L}(Z,E)$, and for every $t \in [a,b]$ a Gateaux differentiable mapping $H_t(\zeta) : E \to Z^*$, measurable on (ζ,t), such that $H_t(0) \in L^2(a,b;Z^*)$ and satisfying

$$\big\|DH_t(L_0 \cdot z)\big\|_{\mathcal{L}(E;Z^*)} \leq \widetilde{g}\big(\|P \cdot z\|_H\big) \quad \forall z \in Z,\ \forall t \in [a,b] \tag{3.134}$$

for some nondecreasing function $\widetilde{g}(s):[0,+\infty)\to(0,+\infty)$, and satisfying

$$\Big\langle h,\Lambda_t\cdot(Q\cdot h)+F_t(T\cdot h)\Big\rangle_{X\times X^*}\geq\Big\langle Q\cdot h,A_t\cdot(Q\cdot h)+H_t\big((L_0\circ Q)\cdot h\big)\Big\rangle_{Z\times Z^*}\qquad\forall h\in X\,,\ \forall t\in[a,b]\,,\tag{3.135}$$

where $A_t\in L^\infty\big(a,b;\mathcal{L}(Z,Z^*)\big)$ is such that $\langle z,A_t\cdot z\rangle_{Z\times Z^*}\geq 0\ \forall z\in Z$, then the function $z(t)$, as above, satisfies the following energy inequality

$$\frac{1}{2}\big\|w(t)\big\|_H^2+\int_a^t\Big\langle z(s),A_s\cdot\big(z(s)\big)+\widetilde{T}\cdot G_s\big(L\cdot z(s)\big)+H_s\big(L_0\cdot z(s)\big)\Big\rangle_{Z\times Z^*}ds\leq\frac{1}{2}\big\|w_0\big\|_H^2\qquad\forall t\in[a,b]\,.\tag{3.136}$$

As a particular case of Theorem 3.5, where $Z=H$, we have the following statement, which is useful in the study of Hyperbolic systems.

Corollary 3.1. *Let $\{X,H,X^*\}$ be an evolution triple with the corresponding inclusion operator $T\in\mathcal{L}(X;H)$ as it was defined in Definition 2.8 together with the corresponding operator $\widetilde{T}\in\mathcal{L}(H;X^*)$ defined as in (2.12). Assume also that the Banach space X is separable. Next let $a,b\in\mathbb{R}$ s.t. $a<b$. Furthermore, for every $t\in[a,b]$ let $\Lambda_t\in L^\infty\big(a,b;\mathcal{L}(H,X^*)\big)$, which satisfies the following positivity condition*

$$\Big\langle h,\Lambda_t\cdot(T\cdot h)\Big\rangle_{X\times X^*}\geq 0\qquad\forall h\in X\,,\ \forall t\in[a,b]\,.\tag{3.137}$$

Next let $F_t(w):H\to X^*$ be a function which is Gateaux differentiable on every $w\in H$ for every $t\in[a,b]$, and satisfies $F_t(0)\in L^2(a,b;X^*)$ and the Lipshitz condition

$$\big\|DF_t(w)\big\|_{\mathcal{L}(H;X^*)}\leq g\big(\|w\|_H\big)\qquad\forall w\in H,\ \forall t\in[a,b]\,,\tag{3.138}$$

for some nondecreasing function $g(s):[0,+\infty)\to(0,+\infty)$. We also assume that $F_t(w)$ is Borel on the pair of variables (w,t) (see Definition 2.2). Moreover, assume that F_t is weak to weak continuous from H to X^* for every fixed t i.e. for every sequence $\{h_n\}_{n=1}^{+\infty}\subset H$ such that $h_n\rightharpoonup h_0$ weakly in H and for every $t\in[a,b]$, we have $F_t(h_n)\rightharpoonup F_t(h_0)$ weakly in X^*. Finally suppose that Λ_t and F_t satisfy the following lower bound condition

$$\Big\langle h,\Lambda_t\cdot(T\cdot h)+F_t(T\cdot h)\Big\rangle_{X\times X^*}\geq-\bar{C}\Big(\big\|T\cdot h\big\|_H^2+1\Big)-\mu(t)\qquad\forall h\in X\,,\ \forall t\in[a,b]\,,\tag{3.139}$$

for nonnegative function $\mu(t)\in L^1(a,b;\mathbb{R})$ and some constant $\bar{C}>0$. Then for every $w_0\in H$ there exists $w(t)\in L^\infty(a,b;H)$, such that $v(t):=\widetilde{T}\cdot\big(w(t)\big)\in W^{1,2}(a,b;X^*)$ and $w(t)$ satisfy the following equation

$$\begin{cases}\frac{dv}{dt}(t)+\Lambda_t\cdot\big(w(t)\big)+F_t\big(w(t)\big)=0\quad\text{for a.e. } t\in(a,b)\,,\\ w(a)=w_0\,,\end{cases}\tag{3.140}$$

where we assume that $w(t)$ is H-weakly continuous on $[a,b]$, as it was stated in Corollary 2.1.

4 Applications

4.1 Notations in the present section

For a $p\times q$ matrix A with ij-th entry a_{ij} we denote by $|A|=\big(\Sigma_{i=1}^p\Sigma_{j=1}^q a_{ij}^2\big)^{1/2}$ the Frobenius norm of A.
For two matrices $A,B\in\mathbb{R}^{p\times q}$ with ij-th entries a_{ij} and b_{ij} respectively, we write $A:B:=\sum_{i=1}^p\sum_{j=1}^q a_{ij}b_{ij}$.
Given a vector valued function $f(x)=\big(f_1(x),\ldots,f_k(x)\big):\Omega\to\mathbb{R}^k$ ($\Omega\subset\mathbb{R}^N$) we denote by $\nabla_x f$ the $k\times N$ matrix with ij-th entry $\frac{\partial f_i}{\partial x_j}$.
For a matrix valued function $F(x):=\{F_{ij}(x)\}:\mathbb{R}^N\to\mathbb{R}^{k\times N}$ we denote by $div\,F$ the $\mathbb{R}^k$-valued

vector field defined by $div\, F := (l_1, \dots, l_k)$ where $l_i = \sum_{j=1}^{N} \frac{\partial F_{ij}}{\partial x_j}$.

For $u = (u_1, \dots, u_p) \in \mathbb{R}^p$ and $v = (v_1, \dots, v_q) \in \mathbb{R}^q$ we denote by $\boldsymbol{u} \otimes \boldsymbol{v}$ the $p \times q$ matrix with ij-th entry $u_i v_j$.

4.2 A general parabolic system in a divergent form

Let $\Psi(A, x, t) : \mathbb{R}_A^{k\times N} \times \mathbb{R}_x^N \times \mathbb{R}_t \to \mathbb{R}$ be a nonnegative measurable function. Moreover assume that $\Psi(A, x, t)$ is C^1 as a function of the first argument A when (x, t) are fixed, which satisfies $\Psi(0, x, t) = 0$ and it is convex by the first argument A when (x, t) are fixed, i.e.

$$\Psi\big(\alpha A_1 + (1-\alpha)A_2, x, t\big) \le \alpha \Psi(A_1, x, t) + (1-\alpha)\Psi(A_2, x, t)$$

for every $\alpha \in [0, 1]$, $A_1, A_2 \in \mathbb{R}^{k\times N}$, $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$. Moreover, we assume that Ψ satisfies the following growth condition

$$(1/C)|A|^q - |g_0(x)| \le \Psi(A, x, t) \le C|A|^q + |g_0(x)| \qquad \forall A \in \mathbb{R}^{k\times N},\ \forall x \in \mathbb{R}^N,\ \forall t \in \mathbb{R}\,, \tag{4.1}$$

where $C > 0$ is some constant, $g_0(x) \in L^1(\mathbb{R}^N, \mathbb{R})$ and $q \in [2, +\infty)$. Next let $\Gamma(A, x, t) : \mathbb{R}_A^{k\times N} \times \mathbb{R}_x^N \times \mathbb{R}_t \to \mathbb{R}^{k\times N}$ be a measurable function. Moreover assume that $\Gamma(A, x, t)$ is C^1 as a function of the first argument A when (x, t) are fixed, which satisfies,

$$\Gamma(0, x, t) \in L^{q^*}\big(\mathbb{R}; L^2(\mathbb{R}^N, \mathbb{R}^{k\times N})\big)\,, \tag{4.2}$$

the following monotonicity condition

$$\sum_{1\le j,n\le N} \sum_{1\le i,m\le k} H_{ij} H_{mn} \frac{\partial \Gamma_{mn}}{\partial A_{ij}}(A, x, t) \ge 0 \qquad \forall H, A \in \mathbb{R}^{k\times N},\ \forall x \in \mathbb{R}^N,\ \forall t \in \mathbb{R}\,, \tag{4.3}$$

and the following growth condition

$$\left|\frac{\partial \Gamma}{\partial A_{ij}}(A, x, t)\right| \le C\,|A|^{q-2} + C$$
$$\forall A \in \mathbb{R}^{k\times N},\ \forall x \in \mathbb{R}^N,\ \forall t \in \mathbb{R}, \quad \forall i \in \{1, \dots, k\},\ \forall j \in \{1, \dots, N\}\,, \tag{4.4}$$

where $C > 0$ is some constant. Finally let $\Xi(B, x, t) : \mathbb{R}_B^k \times \mathbb{R}_x^N \times \mathbb{R}_t \to \mathbb{R}^{k\times N}$ and $\Theta(B, x, t) : \mathbb{R}_B^k \times \mathbb{R}_x^N \times \mathbb{R}_t \to \mathbb{R}^k$ be two measurable functions. Moreover, assume that $\Xi(B, x, t)$ and $\Theta(B, x, t)$ are C^1 as a functions of the first argument B when (x, t) are fixed. We also assume that $\Xi(B, x, t)$ and $\Theta(B, x, t)$ are globally Lipschitz by the first argument B and satisfy

$$\Xi(0, x, t) \in L^{q^*}\big(\mathbb{R}; L^2(\mathbb{R}^N, \mathbb{R}^{k\times N})\big),\ \ \Theta(0, x, t) \in L^{q^*}\big(\mathbb{R}; L^2(\mathbb{R}^N, \mathbb{R}^k)\big)\,. \tag{4.5}$$

Proposition 4.1. *Let $\Psi, \Gamma, \Xi, \Theta$ be as above and let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $2 \le q < +\infty$ and $T_0 > 0$. Then for every $w_0(x) \in L^2(\Omega, \mathbb{R}^k)$ there exists $u(x, t) \in L^q\big(0, T_0; W_0^{1,q}(\Omega, \mathbb{R}^k)\big)$, such that $u(x, t) \in L^\infty\big(0, T_0; L^2(\Omega, \mathbb{R}^k)\big) \cap W^{1,q^*}\big(0, T_0; W^{-1,q^*}(\Omega, \mathbb{R}^k)\big)$, where $q^* := q/(q-1)$, $u(x, t)$ is $L^2(\Omega, \mathbb{R}^k)$-weakly continuous on $[0, T_0]$, $u(x, 0) = w_0(x)$ and $u(x, t)$ is a solution to*

$$\frac{du}{dt}(x, t) = \Theta\big(u(x, t), x, t\big) + div_x\Big(\Xi\big(u(x, t), x, t\big)\Big) +$$
$$div_x\Big(\Gamma\big(\nabla_x u(x, t), x, t\big)\Big) + div_x\Big(D_A\Psi\big(\nabla_x u(x, t), x, t\big)\Big) \qquad \text{in } \Omega \times (0, T_0)\,, \tag{4.6}$$

where

$$D_A\Psi(A, x, t) := \left\{\frac{\partial \Psi}{\partial A_{ij}}(A, x, t)\right\}_{1\le i\le k, 1\le j\le N} \in \mathbb{R}^{k\times N}\,.$$

Moreover if $\Psi(A, x, t)$ is a uniformly convex function by the first argument A then such a solution u is unique.

Proof. Let $X := W_0^{1,q}(\Omega,\mathbb{R}^k)$ (a separable reflexive Banach space), $H := L^2(\Omega,\mathbb{R}^k)$ (a Hilbert space) and $T \in \mathcal{L}(X;H)$ be a usual embedding operator from $W_0^{1,q}(\Omega,\mathbb{R}^k)$ into $L^2(\Omega,\mathbb{R}^k)$. Then T is an injective inclusion with dense image. Furthermore, $X^* = W^{-1,q^*}(\Omega,\mathbb{R}^k)$ where $q^* = q/(q-1)$ and the corresponding operator $\widetilde{T} \in \mathcal{L}(H;X^*)$, defined as in (2.12), is a usual inclusion of $L^2(\Omega,\mathbb{R}^k)$ into $W^{-1,q^*}(\Omega,\mathbb{R}^k)$. Then $\{X,H,X^*\}$ is an evolution triple with the corresponding inclusion operators $T \in \mathcal{L}(X;H)$ and $\widetilde{T} \in \mathcal{L}(H;X^*)$, as it was defined in Definition 2.8. Moreover, by the Theorem about the compact embedding in Sobolev Spaces it is well known that T is a compact operator.

Next, for every $t \in [0,T_0]$ let $\Phi_t(x) : X \to [0,+\infty)$ be defined by

$$\Phi_t(u) := \int_\Omega \Psi\big(\nabla u(x),x,t\big)dx + \frac{k_\Omega}{2}\int_\Omega |u(x)|^2 dx \quad \forall u \in W^{1,q}(\Omega,\mathbb{R}^k) \equiv X\,.$$

where

$$k_\Omega := \begin{cases} 0 & \text{if } \Omega \text{ is bounded}\,,\\ 1 & \text{if } \Omega \text{ is unbounded}\,.\end{cases} \tag{4.7}$$

Then $\Phi_t(x)$ is Gateaux differentiable at every $x \in X$, satisfy $\Phi_t(0) = 0$ and by (4.1) it satisfies the growth condition

$$(1/C)\,\|x\|_X^q - C \le \Phi_t(x) \le C\,\|x\|_X^q + C \quad \forall x \in X,\ \forall t \in [0,T]\,,$$

Furthermore, for every $t \in [0,T_0]$ let $\Lambda_t(x) : X \to X^*$ be defined by

$$\Big\langle \delta, \Lambda_t(u)\Big\rangle_{X\times X^*} := \int_\Omega \Gamma\big(\nabla u(x),x,t\big) : \nabla\delta(x)\,dx \quad \forall u,\delta \in W^{1,q}(\Omega,\mathbb{R}^k) \equiv X\,.$$

Then $\Lambda_t(x) : X \to X^*$ is Gateaux differentiable at every $x \in X$, and, by (4.4) its derivative satisfies the growth condition

$$\|D\Lambda_t(x)\|_{\mathcal{L}(X;X^*)} \le C\,\|x\|_X^{q-2} + C \quad \forall x \in X,\ \forall t \in [0,T_0]\,,$$

for some $C > 0$. Moreover, by (4.3), Λ_t satisfy the following monotonicity conditions

$$\Big\langle h, D\Lambda_t(x)\cdot h\Big\rangle_{X\times X^*} \ge 0 \qquad \forall x,h \in X\ \forall t \in [0,T_0]\,.$$

Finally for every $t \in [0,T_0]$ let $F_t(w) : H \to X^*$ be defined by

$$\Big\langle \delta, F_t(w)\Big\rangle_{X\times X^*} := \int_\Omega \Big\{\Xi\big(w(x),x,t\big) : \nabla\delta(x) - \Big(k_\Omega w(x) + \Theta\big(w(x),x,t\big)\Big)\cdot\delta(x)\Big\}dx$$
$$\forall w \in L^2(\Omega,\mathbb{R}^k) \equiv H,\ \forall \delta \in W^{1,q}(\Omega,\mathbb{R}^k) \equiv X\,. \tag{4.8}$$

Then $F_t(w)$ is Gateaux differentiable at every $w \in H$, and, since Ξ and Θ are Lipshitz functions, the derivative of $F_t(w)$ satisfy the Lipschitz condition

$$\|DF_t(w)\|_{\mathcal{L}(H;X^*)} \le C \quad \forall w \in H,\ \forall t \in [0,T_0]\,, \tag{4.9}$$

for some $C > 0$. Thus all the conditions of Theorem 3.3 are satisfied. Applying this Theorem completes the proof. □

Remark 4.1. If in the framework of Proposition 4.1 we suppose $q = 2$ and that $D_A\Psi(A,x,t)$ and $\Gamma(A,x,t)$ are linear by the first argument A, however we assume that Ω is unbounded, we obtain the similar existence result as in Proposition 4.1, as a consequence of Theorem 3.5 with $Z = X$.

Indeed in the case of unbounded Ω, let $V_j = L^2(\Omega \cap B_{R_j}(0),\mathbb{R}^k)$ for some sequence $R_j \to +\infty$ and set $L_j \in \mathcal{L}(H,V_j)$ by

$$L_j \cdot \big(h(x)\big) := h(x)\llcorner\Omega \cap B_{R_j}(0) \in L^2(\Omega \cap B_{R_j}(0),\mathbb{R}^k) = V_j \qquad \forall h(x) \in L^2(\Omega,\mathbb{R}^k) = H\,.$$

Then by the standard embedding theorems in the Sobolev Spaces the operator $L_j \circ T \in \mathcal{L}(X,V_j)$ is compact for every j. Moreover, if $\{h_n\} \subset H$ is a sequence such that $h_n \rightharpoonup h_0$ weakly in H and $L_j \cdot h_n \to L_j \cdot h_0$ strongly in V_j as $n \to +\infty$ for every j, then we have $h_n \to h_0$ strongly in $L^2_{loc}(\Omega,\mathbb{R}^k)$ and thus, by (4.8) and (4.9) we must have $F_t(h_n) \rightharpoonup F_t(h_0)$ weakly in X^*.

4.3 Parabolic systems in a non-divergent form

Let $\Psi(L,x,t) : \mathbb{R}^k_L \times \mathbb{R}^N_x \times \mathbb{R}_t \to \mathbb{R}$ be a nonnegative measurable function. Moreover, assume that $\Psi(L,x,t)$ is C^1 as a function of the first argument L when (x,t) are fixed, which satisfies $\Psi(0,x,t) = 0$ and it is convex by the first argument L when (x,t) are fixed, i.e.

$$\Psi\big(\alpha L_1 + (1-\alpha)L_2, x, t\big) \leq \alpha\Psi(L_1,x,t) + (1-\alpha)\Psi(L_2,x,t)$$

for every $\alpha \in [0,1]$, $L_1, L_2 \in \mathbb{R}^k$, $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$. Moreover, we assume that Ψ satisfies the following growth condition

$$(1/C)|L|^q - C \leq \Psi(L,x,t) \leq C|L|^q + C \qquad \forall L \in \mathbb{R}^k,\ \forall x \in \mathbb{R}^N,\ \forall t \in \mathbb{R}\,, \tag{4.10}$$

where $C > 0$ is some constant and $q \in [2,+\infty)$. Next let $\Gamma(L,x,t) : \mathbb{R}^k_L \times \mathbb{R}^N_x \times \mathbb{R}_t \to \mathbb{R}^k$ be a measurable function. Moreover, assume that $\Gamma(L,x,t)$ is C^1 as a function of the first argument L when (x,t) are fixed, which satisfies

$$\Gamma(0,x,t) \in L^{q^*}\big(\mathbb{R}; L^2(\mathbb{R}^N,\mathbb{R}^k)\big)\,, \tag{4.11}$$

the following monotonicity condition

$$\sum_{1\leq i,j\leq k} h_i h_j \frac{\partial \Gamma_i}{\partial L_j}(L,x,t) \geq 0 \qquad \forall\, h, L \in \mathbb{R}^k,\ \forall x \in \mathbb{R}^N,\ \forall t \in \mathbb{R}\,, \tag{4.12}$$

and the following growth condition

$$\left|\frac{\partial \Gamma}{\partial L_j}(L,x,t)\right| \;\leq\; C\,|L|^{q-2} + C \qquad \forall L \in \mathbb{R}^k,\ \forall x \in \mathbb{R}^N,\ \forall t \in \mathbb{R},\quad \forall j \in \{1,\ldots,k\}\,. \tag{4.13}$$

Finally let $\Theta(A,L,x,t) : \mathbb{R}^{k\times N}_A \times \mathbb{R}^k_L \times \mathbb{R}^N_x \times \mathbb{R}_t \to \mathbb{R}^k$ be a measurable function. Moreover, assume that $\Theta(A,L,x,t)$ is C^1 as a function of the first two arguments A and L when (x,t) are fixed. We also assume that $\Theta(A,L,x,t)$ is globally Lipschitz by the first two arguments A and L and

$$\Theta(0,0,x,t) \in L^{q^*}\big(\mathbb{R}; L^2(\mathbb{R}^N,\mathbb{R}^k)\big)\,. \tag{4.14}$$

Proposition 4.2. *Let Ψ, Γ, Θ be as above and let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $2 \leq q < +\infty$ and $T_0 > 0$. Then for every $w_0(x) \in W^{1,2}_0(\Omega,\mathbb{R}^k)$ there exists $u(x,t) \in L^q\big(0,T_0; W^{2,q}_{loc}(\Omega,\mathbb{R}^k)\big)$, such that $\Delta_x u(x,t) \in L^q\big(0,T_0; L^q(\Omega,\mathbb{R}^k)\big)$, $u(x,t) \in L^\infty\big(0,T_0; W^{1,2}_0(\Omega,\mathbb{R}^k)\big) \cap W^{1,q^*}\big(0,T_0; L^{q^*}(\Omega,\mathbb{R}^k)\big)$,where $q^* := q/(q-1)$, $u(x,t)$ is $W^{1,2}_0(\Omega,\mathbb{R}^k)$-weakly continuous on $[0,T_0]$, $u(x,0) = w_0(x)$ and $u(x,t)$ is a solution to*

$$\frac{du}{dt}(x,t) = \Theta\big(\nabla_x u(x,t), u(x,t), x, t\big) + \Gamma\big(\Delta_x u(x,t), x, t\big) + \nabla_L \Psi\big(\Delta_x u(x,t), x, t\big) \qquad in\ \ \Omega\times(0,T_0)\,, \tag{4.15}$$

where $\nabla_L \Psi(L,x,t)$ is a partial gradient by the first variable L. Moreover if $\Psi(L,x,t)$ is uniformly convex function by the first argument L then such a solution u is unique.

Proof. Let

$$X := \Big\{u(x) \in W^{1,2}_0(\Omega,\mathbb{R}^k) :\ \Delta u(x) \in L^q(\Omega,\mathbb{R}^k)\Big\}\,, \tag{4.16}$$

for $2 \leq q < +\infty$ endowed with the norm

$$\|u\|_X := \|\Delta u\|_{L^q(\Omega,\mathbb{R}^k)} + \|\nabla u\|_{L^2(\Omega,\mathbb{R}^{k\times N})} \qquad \forall u \in X \subset W^{1,2}_0(\Omega,\mathbb{R}^k)\,. \tag{4.17}$$

Thus X is a separable reflexive Banach space. Next let $H := W^{1,2}_0(\Omega,\mathbb{R}^k)$ endowed with the standard scalar product $< \phi_1, \phi_2 >_{H\times H} = \int_\Omega \nabla\phi_1(x) : \nabla\phi_2(x)\, dx$ (a Hilbert space) and $T \in \mathcal{L}(X;H)$ be a trivial embedding operator from $X \subset W^{1,2}_0(\Omega,\mathbb{R}^k)$ into $H = W^{1,2}_0(\Omega,\mathbb{R}^k)$. Then T is an injective

inclusion with dense image. Moreover, T is a compact operator. In order to follow the definitions above we identify the dual space H^* with H. So in our notations $\big\{W_0^{1,2}(\Omega,\mathbb{R}^k)\big\}^* = W_0^{1,2}(\Omega,\mathbb{R}^k)$ (although in the usual notations $\big\{W_0^{1,2}(\Omega,\mathbb{R}^k)\big\}^*$ identified with the isomorphic space $W^{-1,2}(\Omega,\mathbb{R}^k)$). Next define $S\in\mathcal{L}\big(L^{q*}(\Omega,\mathbb{R}^k),X^*\big)$ by the formula

$$\Big\langle\delta,S\cdot h\Big\rangle_{X\times X^*}=-\int_\Omega h(x)\cdot\Delta\delta(x)\,dx\qquad\forall\delta\in X,\ \forall h\in L^{q^*}(\Omega,\mathbb{R}^k)\,. \tag{4.18}$$

Then, since for every $\phi\in L^q(\Omega,\mathbb{R}^k)$ there exists unique $\delta_\phi\in X$ such that $\Delta\delta_\phi=\phi$ we deduce that S is an injective inclusion (i.e. $\ker S=0$). For the corresponding operator $\widetilde{T}\in\mathcal{L}(H;X^*)$, by (2.12) and (4.18) we must have

$$\big\langle u,\widetilde{T}\cdot w\big\rangle_{X\times X^*}:=\big\langle T\cdot u,w\big\rangle_{H\times H}=\int_\Omega\nabla u(x):\nabla w(x)\,dx=-\int_\Omega w(x)\cdot\Delta u(x)\,dx=\Big\langle u,S\cdot(L\cdot w)\Big\rangle_{X\times X^*}$$
$$\text{for every } w\in H \text{ and } u\in X\,, \tag{4.19}$$

where L is a trivial inclusion of $W_0^{1,2}(\Omega,\mathbb{R}^k)$ into $L^{q^*}(\Omega,\mathbb{R}^k)$ ($q^*\leq 2$). So

$$\widetilde{T}=S\circ L\,. \tag{4.20}$$

Then $\{X,H,X^*\}$ is an evolution triple with the corresponding inclusion operators $T\in\mathcal{L}(X;H)$ and $\widetilde{T}\in\mathcal{L}(H;X^*)$, as it was defined in Definition 2.8.

Next, for every $t\in[0,T_0]$ let $\Phi_t(x):X\to[0,+\infty)$ be defined by

$$\Phi_t(u):=\int_\Omega\Big(\Psi\big(\Delta u(x),x,t\big)+\frac{1}{2}\big|\nabla u(x)\big|^2\Big)dx\quad\forall u\in X\,.$$

Then $\Phi_t(x)$ is Gateaux differentiable at every $x\in X$, satisfies $\Phi_t(0)=0$ and it satisfies the growth condition

$$(1/C)\,\|x\|_X^q-C\leq\Phi_t(x)\leq C\,\|x\|_X^q+C\quad\forall x\in X,\ \forall t\in[0,T_0]\,.$$

Furthermore, for every $t\in[0,T_0]$ let $\Lambda_t(x):X\to X^*$ be defined by

$$\Big\langle\delta,\Lambda_t(u)\Big\rangle_{X\times X^*}:=\int_\Omega\Gamma\big(\Delta u(x),x,t\big)\cdot\Delta\delta(x)\,dx\quad\forall u,\delta\in X\,,$$

i.e.

$$\Lambda_t(u)=-S\cdot\Big(\Gamma\big(\Delta u(x),x,t\big)\Big)\quad\forall u\in X\,. \tag{4.21}$$

Then $\Lambda_t(x):X\to X^*$ is Gateaux differentiable at every $x\in X$, and, by (4.4) its derivative satisfies the growth condition

$$\|D\Lambda_t(x)\|_{\mathcal{L}(X;X^*)}\leq C\,\|x\|_X^{q-2}+C\quad\forall x\in X,\ \forall t\in[0,T_0]\,,$$

for some $C>0$. Moreover, by (4.3), Λ_t satisfies the following monotonicity conditions

$$\Big\langle h,D\Lambda_t(x)\cdot h\Big\rangle_{X\times X^*}\geq 0\qquad\forall x,h\in X\ \forall t\in[0,T_0]\,.$$

Finally for every $t\in[0,T_0]$ let $F_t(w):H\to X^*$ be defined by

$$\Big\langle\delta,F_t(w)\Big\rangle_{X\times X^*}:=\int_\Omega\Big(\Theta\big(\nabla w(x),w(x),x,t\big)+w(x)\Big)\cdot\Delta\delta(x)dx\quad\forall w\in W_0^{1,2}(\Omega,\mathbb{R}^k)\equiv H,\ \forall\delta\in X\,,$$

i.e.

$$F_t(w)=-S\cdot\Big(\Theta\big(\nabla w(x),w(x),x,t\big)+w(x)\Big)\quad\forall w\in H\,. \tag{4.22}$$

Then $F_t(w)$ is Gateaux differentiable at every $w\in H$, and, since Θ is a Lipshitz function, the derivative of $F_t(w)$ satisfies Lipschitz condition

$$\|DF_t(w)\|_{\mathcal{L}(H;X^*)}\leq C\quad\forall w\in H,\ \forall t\in[0,T_0]\,. \tag{4.23}$$

Thus all the conditions of Theorem 3.3 are satisfied. Applying this Theorem and (4.18), we obtain that for every $w_0(x) \in W_0^{1,2}(\Omega,\mathbb{R}^k)$ there exists $u(x,t) \in L^q\big(0,T_0;W_{loc}^{2,q}(\Omega,\mathbb{R}^k)\big)$, such that $u(x,t) \in L^\infty\big(0,T_0;W_0^{1,2}(\Omega,\mathbb{R}^k)\big)$,where $q^* := q/(q-1)$, $u(x,t)$ is $W_0^{1,2}(\Omega,\mathbb{R}^k)$-weakly continuous on $[0,T_0]$, $u(x,0) = w_0(x)$ and $u(x,t)$ is a solution to

$$\frac{dv}{dt}(t) + \Lambda_t\big(u(t)\big) + F_t\big(w(t)\big) + D\Phi_t\big(u(t)\big) = 0 \quad \text{for a.e. } t \in (0,T_0)\,. \tag{4.24}$$

Thus, by (4.24), (4.18), (4.20), (4.21), (4.22) and Lemma 2.1 we infer that $u(x,t) \in W^{1,q^*}\big(0,T_0;L^{q^*}(\Omega,\mathbb{R}^k)\big)$ and

$$\int\limits_\Omega \Bigg\{ -\frac{du}{dt}(x,t) + \Theta\big(\nabla_x u(x,t),u(x,t),x,t\big) \\ + \Gamma\big(\Delta_x u(x,t),x,t\big) + \nabla_L\Psi\big(\Delta_x u(x,t),x,t\big)\Bigg\} \cdot \Delta\delta(x)\,dx = 0 \\ \forall t \in (0,T_0),\ \forall\delta(x) \in X\,. \tag{4.25}$$

Therefore

$$\frac{du}{dt}(x,t) = \Theta\big(\nabla_x u(x,t),u(x,t),x,t\big) + \Gamma\big(\Delta_x u(x,t),x,t\big) + \nabla_L\Psi\big(\Delta_x u(x,t),x,t\big) \\ \forall(x,t) \in \Omega\times(0,T_0)\,, \tag{4.26}$$

and the result follows. □

4.4 Hyperbolic systems of second order

Proposition 4.3. *Let $\Omega \subset \mathbb{R}^N$ be an open set and $T_0 > 0$. Furthermore, let $\Xi(L,x,t) : \mathbb{R}^k_L \times \mathbb{R}^N_x \times \mathbb{R}_t \to \mathbb{R}^{k\times N}$, $\Upsilon(L,x,t) : \mathbb{R}^k_L \times \mathbb{R}^N_x \times \mathbb{R}_t \to \mathbb{R}^k$ and $\Theta(L,x,t) : \mathbb{R}^k_L \times \mathbb{R}^N_x \times \mathbb{R}_t \to \mathbb{R}^k$ be measurable functions. Moreover, assume that $\Xi(L,x,t)$, $\Upsilon(L,x,t)$ and $\Theta(L,x,t)$ are C^1 as a functions of the first argument L when (x,t) are fixed. We also assume that $\Upsilon(L,x,t)$ $\nabla_x\Upsilon(L,x,t)$, $\Theta(L,x,t)$, $\Xi(L,x,t)$ and $\nabla_x\Xi(L,x,t)$ are globally Lipschitz by the first argument L, $\Upsilon(L,x,t)$ is globally Lipschitz by the last argument t, $\Theta(0,x,t) \in L^2\big(\mathbb{R};L^2(\mathbb{R}^N,\mathbb{R}^k)\big)$, $\Xi(0,x,t) \in L^2\big(\mathbb{R};W^{1,2}(\mathbb{R}^N,\mathbb{R}^{k\times N})\big)$ and $\Upsilon(0,x,t) \in L^2\big(\mathbb{R};W_0^{1,2}(\Omega,\mathbb{R}^k)\big)$. Then for every $w_0(x) \in W_0^{1,2}(\Omega,\mathbb{R}^k)$ and $h_0(x) \in L^2(\Omega,\mathbb{R}^k)$ there exists $u(x,t) \in L^\infty\big(0,T_0;W_0^{1,2}(\Omega,\mathbb{R}^k)\big)$ such that $\frac{du}{dt}(x,t) \in L^\infty\big(0,T_0;L^2(\Omega,\mathbb{R}^k)\big) \cap W^{1,2}\big(0,T_0;W^{-1,2}(\Omega,\mathbb{R}^k)\big)$, $u(x,t)$ is $W_0^{1,2}(\Omega,\mathbb{R}^k)$-weakly continuous on $[0,T_0]$, $\frac{du}{dt}(x,t)$ is $L^2(\Omega,\mathbb{R}^k)$-weakly continuous on $[0,T_0]$, $u(x,0) = w_0(x)$, $\frac{du}{dt}(x,0) = h_0(x)$ and $u(x,t)$ is a solution to*

$$\frac{d^2u}{dt^2}(x,t) - \Delta_x u(x,t) + \partial_t\big\{\Upsilon\big(u(x,t),x,t\big)\big\} + div_x\big\{\Xi\big(u(x,t),x,t\big)\big\} + \Theta\big(u(x,t),x,t\big) = 0 \\ \text{in } \Omega\times(0,T_0)\,. \tag{4.27}$$

Proof. Let $X_0 := \big\{\varphi \in W_0^{1,2}(\Omega,\mathbb{R}^k) \cap W_{loc}^{2,2}(\Omega,\mathbb{R}^k) : \Delta\varphi \in L^2(\Omega,\mathbb{R}^k)\big\}$ endowed with the norm

$$\|\varphi\|_{X_0} := \sqrt{\|\Delta\varphi\|^2_{L^2(\Omega,\mathbb{R}^k)} + \|\nabla\varphi\|^2_{L^2(\Omega,\mathbb{R}^{k\times N})} + \|\varphi\|^2_{L^2(\Omega,\mathbb{R}^k)}} \qquad \forall\varphi \in X_0 \subset W_{loc}^{2,2}(\Omega,\mathbb{R}^k) \cap W_0^{1,2}(\Omega,\mathbb{R}^k)\,, \tag{4.28}$$

Then X_0 is a separable reflexive Banach space. Next let $H_0 := W_0^{1,2}(\Omega,\mathbb{R}^k)$ endowed with the standard scalar product $<\phi_1,\phi_2>_{H\times H} = \int_\Omega \big(\nabla\phi_1(x) : \nabla\phi_2(x) + \phi_1(x)\cdot\phi_2(x)\big)\,dx$ (a Hilbert space) and $\mathcal{T}_0 \in \mathcal{L}(X_0;H_0)$ be a trivial embedding operator from $X_0 \subset W_0^{1,2}(\Omega,\mathbb{R}^k)$ into $H_0 = W_0^{1,2}(\Omega,\mathbb{R}^k)$. Then $\mathcal{T}_0$ is an injective inclusion with dense image. As before, in out notations, $\big\{W_0^{1,2}(\Omega,\mathbb{R}^k)\big\}^* = W_0^{1,2}(\Omega,\mathbb{R}^k)$ (although in the usual notations $\big\{W_0^{1,2}(\Omega,\mathbb{R}^k)\big\}^*$ identified with the isomorphic space $W^{-1,2}(\Omega,\mathbb{R}^k)$). Next, define $S_0 \in \mathcal{L}\big(L^2(\Omega,\mathbb{R}^k),X_0^*\big)$ by

$$\big\langle\delta,S_0\cdot h\big\rangle_{X_0\times X_0^*} = \int_\Omega \Big(\delta(x) - \Delta\delta(x)\Big)\cdot h(x)\,dx \qquad \forall\delta \in X_0,\ \forall h \in L^2(\Omega,\mathbb{R}^k)\,. \tag{4.29}$$

Then, since for every $\phi \in L^2(\Omega, \mathbb{R}^k)$ there exists unique $\delta_\phi \in X_0$ such that $(\Delta\delta_\phi - \delta_\phi) = \phi$ we deduce that S_0 is injective inclusion (i.e. $\ker S_0 = 0$). As before, $\{X_0, H_0, X_0^*\}$ is an evolution triple with the corresponding inclusion operators $\mathcal{T}_0 \in \mathcal{L}(X_0; H_0)$ and $\widetilde{\mathcal{T}}_0 \in \mathcal{L}(H_0; X_0^*)$, as it was defined in Definition 2.8 by

$$\left\langle \delta, \widetilde{\mathcal{T}}_0 \cdot \varphi \right\rangle_{X_0 \times X_0^*} := \left\langle \mathcal{T}_0 \cdot \delta, \varphi \right\rangle_{H_0 \times H_0} \qquad \text{for every } \varphi \in H_0 \text{ and } \delta \in X_0\,. \tag{4.30}$$

However,

$$\left\langle \mathcal{T}_0 \cdot \delta, \varphi \right\rangle_{H_0 \times H_0} = \int_\Omega \Big(\nabla\delta(x) : \nabla\varphi(x) + \delta(x) \cdot \varphi(x)\Big) dx = \\ \int_\Omega \Big(\delta(x) - \Delta\delta(x)\Big) \cdot \varphi(x) dx = \left\langle \delta, (S_0 \circ L) \cdot \varphi \right\rangle_{X_0 \times X_0^*} \qquad \text{for every } \varphi \in H_0 \text{ and } \delta \in X_0\,, \tag{4.31}$$

where $L \in \mathcal{L}\big(W_0^{1,2}(\Omega, \mathbb{R}^k), L^2(\Omega, \mathbb{R}^k)\big)$ is a trivial inclusion of $W_0^{1,2}(\Omega, \mathbb{R}^k)$ into $L^2(\Omega, \mathbb{R}^k)$. Thus plugging (4.31) into (4.30) we obtain

$$\widetilde{\mathcal{T}}_0 \cdot \varphi = S_0 \cdot (L \cdot \varphi) \qquad \text{for every } \varphi \in H_0\,. \tag{4.32}$$

Next, as in the proof of Proposition 4.1, let $X_1 := W_0^{1,2}(\Omega, \mathbb{R}^k)$, $H_1 := L^2(\Omega, \mathbb{R}^k)$ and $T_1 \in \mathcal{L}(X_1; H_1)$ be a usual embedding operator from $W_0^{1,2}(\Omega, \mathbb{R}^k)$ into $L^2(\Omega, \mathbb{R}^k)$. Then T_1 is an injective inclusion with dense image. Furthermore, $X_1^* = W^{-1,2}(\Omega, \mathbb{R}^k)$ and the corresponding operator $\widetilde{T}_1 \in \mathcal{L}(H_1; X_1^*)$, defined as in (2.12), is a usual inclusion of $L^2(\Omega, \mathbb{R}^k)$ into $W^{-1,2}(\Omega, \mathbb{R}^k)$. Thus $\{X_1, H_1, X_1^*\}$ is another evolution triple with the corresponding inclusion operators $T_1 \in \mathcal{L}(X_1; H_1)$ and $\widetilde{T}_1 \in \mathcal{L}(H_1; X_1^*)$, as it was defined in Definition 2.8. Finally set

$$X := \Big\{\big(u(x), v(x)\big) :\ u(x) : \Omega \to \mathbb{R}^k,\ v(x) : \Omega \to \mathbb{R}^k \\ u(x) \in X_0 \subset W_{loc}^{2,2}(\Omega, \mathbb{R}^k) \cap W_0^{1,2}(\Omega, \mathbb{R}^k),\ v(x) \in X_1 \equiv W_0^{1,2}(\Omega, \mathbb{R}^k)\Big\}\,. \tag{4.33}$$

In this space we consider the norm

$$\|z\|_X := \sqrt{\|u\|_{X_0}^2 + \|v\|_{X_1}^2} = \sqrt{\|\Delta u\|^2_{L^2(\Omega,\mathbb{R}^k)} + \|u\|^2_{W_0^{1,2}(\Omega,\mathbb{R}^k)} + \|v\|^2_{W_0^{1,2}(\Omega,\mathbb{R}^k)}} \qquad \forall z = (u, v) \in X\,. \tag{4.34}$$

Thus X is a separable reflexive Banach space. Next set

$$H := \Big\{\big(u(x), v(x)\big) :\ u(x) : \Omega \to \mathbb{R}^k,\ v(x) : \Omega \to \mathbb{R}^k \\ u(x) \in H_0 \equiv W_0^{1,2}(\Omega, \mathbb{R}^k),\ v(x) \in H_1 \equiv L^2(\Omega, \mathbb{R}^k)\Big\}\,. \tag{4.35}$$

In this space we consider the scalar product

$$< z_1, z_2 >_{H \times H} := < u_1, u_2 >_{H_0 \times H_0} + < v_1, v_2 >_{H_1 \times H_1} \\ = \int_\Omega \big\{\nabla u_1(x) : \nabla u_2(x) + u_1(x) \cdot u_2(x) + v_1(x) \cdot v_2(x)\big\} dx \qquad \forall z_1 = (u_1, v_1), z_2 = (u_2, v_2) \in H\,. \tag{4.36}$$

Then H is a Hilbert space. Furthermore, consider $T \in \mathcal{L}(X, H)$ by

$$T \cdot z = \big(\mathcal{T}_0 \cdot u, T_1 \cdot v\big) \qquad \forall z = (u, v) \in X\,. \tag{4.37}$$

Thus T is an injective inclusion with dense image. Furthermore,

$$X^* := \Big\{\big(u, v\big) :\ u \in X_0^*,\ v \in X_1^* \equiv W^{-1,2}(\Omega, \mathbb{R}^k)\Big\}\,, \tag{4.38}$$

where

$$\left\langle\delta,h\right\rangle_{X\times X^*}=\left\langle\delta_0,h_0\right\rangle_{X_0\times X_0^*}+\left\langle\delta_1,h_1\right\rangle_{X_1\times X_1^*}\qquad\forall\delta=(\delta_0,\delta_1)\in X,\ \forall h=(h_0,h_1)\in X^*\,,\tag{4.39}$$

and

$$\|z\|_{X^*}:=\Big(\|u\|^2_{X_0^*}+\|v\|^2_{X_1^*}\Big)^{1/2}\qquad\forall z=(u,v)\in X^*\,.\tag{4.40}$$

Moreover, the corresponding operator $\widetilde{T}\in\mathcal{L}(H;X^*)$, defined as in (2.12), is defined by

$$\widetilde{T}\cdot z=\big(\widetilde{\mathcal{T}}_0\cdot u,\widetilde{T}_1\cdot v\big)\qquad\forall z=(u,v)\in H\,.\tag{4.41}$$

Thus $\{X,H,X^*\}$ is an evolution triple with the corresponding inclusion operators $T\in\mathcal{L}(X;H)$ and $\widetilde{T}\in\mathcal{L}(H;X^*)$, as it was defined in Definition 2.8.

Next let $\Lambda\in\mathcal{L}(H,X^*)$ be defined by by

$$\Lambda\cdot z:=(S_0\cdot v,\Delta u-u)\qquad\forall z=(u,v)\in H\ \ (\text{i.e. } u\in W_0^{1,2}(\Omega,\mathbb{R}^k),\ v\in L^2(\Omega,\mathbb{R}^k))\tag{4.42}$$

Then using (4.39) and (4.29) we deduce

$$\Big\langle h,\Lambda\cdot(T\cdot h)\Big\rangle_{X\times X^*}=\Big\langle u,S_0\cdot(T_1\cdot v)\Big\rangle_{X_0\times X_0^*}+\Big\langle v,\Delta(\mathcal{T}_0\cdot u)-\mathcal{T}_0\cdot u\Big\rangle_{X_1\times X_1^*}$$
$$=\int_\Omega v(x)\cdot\Big(u(x)-\Delta u(x)\Big)dx-\int_\Omega\Big(\nabla v(x):\nabla u(x)+v(x)\cdot u(x)\Big)\,dx=0\qquad\forall h=(u,v)\in X\,.\tag{4.43}$$

Furthermore, for $t\in[0,T_0]$ let $F_t(z):H\to H$ be a function defined by

$$F_t(z)\ :=\ \Big(\Upsilon\big(u(x),x,t\big),u(x)\ -\ \Theta\big(u(x),x,t\big)\ -\ div_x\Xi\big(u(x),x,t\big)\Big)\qquad\forall z\ =\ (u,v)\ \in\ H\,,\tag{4.44}$$

(We have $\Upsilon(u(x),x,t)\in W_0^{1,2}(\Omega,\mathbb{R}^k)$ for a.e. t), i.e.

$$\Big\langle F_t(z),z_0\Big\rangle_{H\times H}=\int_\Omega\Big(\nabla_x\big\{\Upsilon\big(u(x),x,t\big)\big\}:\nabla u_0(x)+\Upsilon\big(u(x),x,t\big)\cdot u_0(x)\Big)dx+$$
$$\int_\Omega\Big\{u(x)-\Theta\big(u(x),x,t\big)-div_x\Xi\big(u(x),x,t\big)\Big\}\cdot v_0(x)dx$$
$$\forall z=(u,v)\in H,\ \forall z_0=(u_0,v_0)\in H\,.\tag{4.45}$$

Then it satisfies the following conditions

$$\|F_t(z)\|_H\le C\|z\|_H+f(t)\qquad\forall z\in H,\ \forall t\in[0,T_0]\,,\tag{4.46}$$

and

$$\|\widetilde{T}\circ DF_t(z)\|_{\mathcal{L}(H;X^*)}\le C\qquad\forall z\in H,\ \forall t\in[0,T_0]\,,\tag{4.47}$$

for some $C>0$ and some $f(t)\in L^2(0,T_0;\mathbb{R})$. Moreover, for bounded Ω, since the embedding of $W_0^{1,2}(\Omega,\mathbb{R}^k)$ into $L^2(\Omega,\mathbb{R}^k)$ is compact we obtain that F_t is weak to weak continuous on H. If we assume Ω to be unbounded then, for every $\Omega'\subset\subset\Omega$, F_t is weak to weak continuous, as a mapping defined on H with the valued functions, restricted to the smaller set Ω'. Therefore, since Ω' is arbitrary, using (4.46) we deduce that in any case F_t is weak to weak continuous on H. Then all the conditions of Corollary 3.1 satisfied and by this Corollary for every $w_0\in W_0^{1,2}(\Omega,\mathbb{R}^k)$ and every $h_0\in L^2(\Omega,\mathbb{R}^k)$ there exists $\zeta(t)\in L^\infty(0,T_0;H)$, such that $\xi(t):=\widetilde{T}\cdot\big(\zeta(t)\big)\in W^{1,2}(0,T_0;X^*)$ and $\zeta(t)$ satisfies the following equation

$$\begin{cases}\frac{d\xi}{dt}(t)+\Lambda\cdot\big(\zeta(t)\big)+\widetilde{T}\cdot F_t\big(\zeta(t)\big)=0\quad\text{for a.e. } t\in(0,T_0)\,,\\ \zeta(0)=\Big(w_0(x),\ -h_0(x)-\Upsilon\big(w_0(x),x,0\big)\Big)\,,\end{cases}\tag{4.48}$$

where we assume that $\zeta(t)$ is H-weakly continuous on $[0,T_0]$, as it was stated in Corollary 2.1. We can rewrite (4.48) as follows. Let $\big(u(x,t),v(x,t)\big)=\zeta(t)$. Then by (4.48), (4.37), (4.42), (4.45), (4.32) and Lemma 2.1, $u(x,t)\in L^\infty\big(0,T_0;W_0^{1,2}(\Omega,\mathbb{R}^k)\big)\cap W^{1,2}\big(0,T_0;L^2(\Omega,\mathbb{R}^k)\big)$, $v(x,t)\in L^\infty\big(0,T_0;L^2(\Omega,\mathbb{R}^k)\big)\cap W^{1,2}\big(0,T_0;W^{-1,2}(\Omega,\mathbb{R}^k)\big)$, $u(x,t)$ is $W_0^{1,2}(\Omega,\mathbb{R}^k)$-weakly continuous on $[0,T_0]$, $v(x,t)$ is $L^2(\Omega,\mathbb{R}^k)$-weakly continuous on $[0,T_0]$, $u(x,0)=w_0(x)$, $v(x,0)=-h_0(x)-\Upsilon\big(w_0(x),x,0\big)$ and in $\Omega\times(0,T_0)$ $\big(u(x,t),v(x,t)\big)$ solves

$$\begin{cases}\frac{du}{dt}(x,t)+v(x,t)+\Upsilon\big(u(x,t),x,t\big)=0\,,\\ \frac{dv}{dt}(x,t)+\Delta_x u(x,t)-\Theta\big(u(x,t),x,t\big)-div_x\Xi\big(u(x,t),x,t\big)=0\,.\end{cases}\tag{4.49}$$

Thus in particular $\frac{du}{dt}(x,t)\in L^\infty\big(0,T_0;L^2(\Omega,\mathbb{R}^k)\big)\cap W^{1,2}\big(0,T_0;W^{-1,2}(\Omega,\mathbb{R}^k)\big)$ and $\frac{du}{dt}(x,0)=h_0(x)$. Moreover, differentiating the equality $v(x,t)=-\frac{du}{dt}(x,t)-\Upsilon\big(u(x,t),x,t\big)$ by the argument t and inserting it into the second equation in (4.49) we finally deduce (4.27). $\square$

4.5 Schrödinger type nonlinear systems

Proposition 4.4. *Let $\Omega\subset\mathbb{R}^N$ be an open set and $T_0>0$. Furthermore, let $\Theta(a,b,x,t):\mathbb{R}^k_a\times\mathbb{R}^k_b\times\mathbb{R}^N_x\times\mathbb{R}_t\to\mathbb{R}^k$ and $\Xi(a,b,x,t):\mathbb{R}^k_a\times\mathbb{R}^k_b\times\mathbb{R}^N_x\times\mathbb{R}_t\to\mathbb{R}^k$ be measurable functions. Moreover, assume that $\Theta(a,b,x,t)$ and $\Xi(a,b,x,t)$ are C^1 as a functions of the first two arguments a and b when (x,t) are fixed. We also assume that $\Theta(a,b,x,t)$, $\nabla_x\Theta(a,b,x,t)$, $\Xi(a,b,x,t)$ and $\nabla_x\Xi(a,b,x,t)$ are globally Lipschitz by the first two arguments a and b, and $\Theta(0,0,x,t)\in L^2\big(\mathbb{R};W_0^{1,2}(\Omega,\mathbb{R}^k)\big)$ and $\Xi(0,0,x,t)\in L^2\big(\mathbb{R};W_0^{1,2}(\Omega,\mathbb{R}^k)\big)$. Then for every $w_0(x)\in W_0^{1,2}(\Omega,\mathbb{R}^k)$ and $h_0(x)\in W_0^{1,2}(\Omega,\mathbb{R}^k)$ there exists $u(x,t)\in L^\infty\big(0,T_0;W_0^{1,2}(\Omega,\mathbb{R}^k)\big)\cap W^{1,2}\big(0,T_0;W^{-1,2}(\Omega,\mathbb{R}^k)\big)$ and $v(x,t)\in L^\infty\big(0,T_0;W_0^{1,2}(\Omega,\mathbb{R}^k)\big)\cap W^{1,2}\big(0,T_0;W^{-1,2}(\Omega,\mathbb{R}^k)\big)$, $u(x,t)$ and $v(x,t)$ are $W_0^{1,2}(\Omega,\mathbb{R}^k)$-weakly continuous on $[0,T_0]$, $u(x,0)=w_0(x)$, $v(x,0)=h_0(x)$ and $\big(u(x,t),v(x,t)\big)$ is a solution to*

$$\begin{cases}\frac{du}{dt}(x,t)-\Delta_x v(x,t)+\Theta\big(u(x,t),v(x,t),x,t\big)=0 & in\ \Omega\times(0,T_0)\,,\\ \frac{dv}{dt}(x,t)+\Delta_x u(x,t)+\Xi\big(u(x,t),v(x,t),x,t\big)=0 & in\ \Omega\times(0,T_0)\,.\end{cases}\tag{4.50}$$

Proof. Let $X_0:=\big\{\varphi\in W_0^{1,2}(\Omega,\mathbb{R}^k)\cap W^{3,2}_{loc}(\Omega,\mathbb{R}^k):\Delta\varphi\in W_0^{1,2}(\Omega,\mathbb{R}^k)\big\}$ endowed with the norm

$$\|\varphi\|_{X_0}:=\sqrt{\|\nabla\Delta\varphi\|^2_{L^2(\Omega,\mathbb{R}^{k\times N})}+\|\Delta\varphi\|^2_{L^2(\Omega,\mathbb{R}^k)}+\|\nabla\varphi\|^2_{L^2(\Omega,\mathbb{R}^{k\times N})}+\|\varphi\|^2_{L^2(\Omega,\mathbb{R}^k)}}\\ \forall\varphi\in X_0\subset W_0^{1,2}(\Omega,\mathbb{R}^k)\cap W^{3,2}_{loc}(\Omega,\mathbb{R}^k)\,.\tag{4.51}$$

So X_0 is a separable reflexive Banach space (in fact it is a Hilbert space). Next let $H_0:=W_0^{1,2}(\Omega,\mathbb{R}^k)$ endowed with the standard scalar product $<\phi_1,\phi_2>_{H\times H}=\int_\Omega\big(\nabla\phi_1(x):\nabla\phi_2(x)+\phi_1(x)\cdot\phi_2(x)\big)dx$ (a Hilbert space) and $\mathcal{T}_0\in\mathcal{L}(X_0;H_0)$ be a trivial embedding operator from $X_0\subset W_0^{1,2}(\Omega,\mathbb{R}^k)$ into $H_0=W_0^{1,2}(\Omega,\mathbb{R}^k)$. Then $\mathcal{T}_0$ is an injective inclusion with dense image. As before, in out notations, $\big\{W_0^{1,2}(\Omega,\mathbb{R}^k)\big\}^*=W_0^{1,2}(\Omega,\mathbb{R}^k)$.

Next, clearly, for every $h\in W^{-1,2}(\Omega,\mathbb{R}^k)$, there exists unique $H_h\in W_0^{1,2}(\Omega,\mathbb{R}^k)$ such that $\Delta H_h-H_h=h$. Then define $S_0\in\mathcal{L}\big(W^{-1,2}(\Omega,\mathbb{R}^k),X_0^*\big)$ by

$$\big\langle\delta,S_0\cdot h\big\rangle_{X_0\times X_0^*}=\int_\Omega\Big\{\Big((\nabla\Delta)\delta(x)-\nabla\delta(x)\Big):\nabla H_h(x)+\Big((\Delta\delta(x)-\delta(x)\Big)\cdot H_h(x)\Big\}dx\\ \forall\delta\in X_0,\ \forall h\in W^{-1,2}(\Omega,\mathbb{R}^k)\,.\tag{4.52}$$

Then, since for every $\phi\in W_0^{1,2}(\Omega,\mathbb{R}^k)$ there exists unique $\delta_\phi\in X_0$ such that $\Delta\delta_\phi-\delta_\phi=\phi$ we deduce that S_0 is injective inclusion (i.e. $\ker S_0=0$). As before, $\{X_0,H_0,X_0^*\}$ is an evolution triple with the corresponding inclusion operators $\mathcal{T}_0\in\mathcal{L}(X_0;H_0)$ and $\widetilde{\mathcal{T}}_0\in\mathcal{L}(H_0;X_0^*)$, as it was defined in Definition 2.8 by

$$\big\langle\delta,\widetilde{\mathcal{T}}_0\cdot\varphi\big\rangle_{X_0\times X_0^*}:=\big\langle\mathcal{T}_0\cdot\delta,\varphi\big\rangle_{H_0\times H_0}\qquad\text{for every }\varphi\in H_0\text{ and }\delta\in X_0\,.\tag{4.53}$$

However,

$$\left\langle \mathcal{T}_0\cdot\delta,\varphi\right\rangle_{H_0\times H_0}=\int_\Omega\Big(\nabla\delta(x):\nabla\varphi(x)+\delta(x)\cdot\varphi(x)\Big)dx=$$
$$\int_\Omega\Big(\delta(x)-\Delta\delta(x)\Big)\cdot\varphi(x)\,dx=\int_\Omega\Big(\delta(x)-\Delta\delta(x)\Big)\cdot\Big(\Delta H_{L\cdot\varphi}(x)-H_{L\cdot\varphi}(x)\Big)dx=$$
$$\int_\Omega\bigg\{\Big((\nabla\Delta)\delta(x)-\nabla\delta(x)\Big):\nabla H_{L\cdot\varphi}(x)+\Big((\Delta\delta(x)-\delta(x)\Big)\cdot H_{L\cdot\varphi}(x)\bigg\}dx$$
$$=\left\langle\delta,(S_0\circ L)\cdot\varphi\right\rangle_{X_0\times X_0^*}\qquad\text{for every }\varphi\in H_0\text{ and }\delta\in X_0\,,\quad(4.54)$$

where $L\in\mathcal{L}\big(W_0^{1,2}(\Omega,\mathbb{R}^k),W^{-1,2}(\Omega,\mathbb{R}^k)\big)$ is a trivial inclusion of $W_0^{1,2}(\Omega,\mathbb{R}^k)$ in $W^{-1,2}(\Omega,\mathbb{R}^k)$. Thus plugging (4.54) into (4.53) we obtain

$$\widetilde{\mathcal{T}}_0\cdot\varphi=S_0\cdot(L\cdot\varphi)\qquad\text{for every }\varphi\in H_0\,.\qquad(4.55)$$

Next set

$$X:=\Big\{\big(u(x),v(x)\big):\ u(x):\Omega\to\mathbb{R}^k,\ v(x):\Omega\to\mathbb{R}^k,\ u(x)\in X_0,\ v(x)\in X_0\Big\}\,.\qquad(4.56)$$

In this space we consider the norm

$$\|z\|_X:=\sqrt{\|u\|_{X_0}^2+\|v\|_{X_0}^2}\qquad\forall z=(u,v)\in X\,.\qquad(4.57)$$

Thus X is a separable reflexive Banach space. Next set

$$H:=\Big\{\big(u(x),v(x)\big):\ u(x):\Omega\to\mathbb{R}^k,\ v(x):\Omega\to\mathbb{R}^k,\ u(x)\in H_0,\ v(x)\in H_0\Big\}\,.\qquad(4.58)$$

In this space we consider the scalar product

$$<z_1,z_2>_{H\times H}:=<u_1,u_2>_{H_0\times H_0}+<v_1,v_2>_{H_0\times H_0}=$$
$$\int_\Omega\Big\{\nabla u_1(x):\nabla u_2(x)+u_1(x)\cdot u_2(x)+\nabla v_1(x):\nabla v_2(x)+v_1(x)\cdot v_2(x)\Big\}dx$$
$$\forall z_1=(u_1,v_1),z_2=(u_2,v_2)\in H\,.\quad(4.59)$$

Then H is a Hilbert space. Furthermore, consider $T\in\mathcal{L}(X,H)$ by

$$T\cdot z=\big(\mathcal{T}_0\cdot u,\mathcal{T}_0\cdot v\big)\qquad\forall z=(u,v)\in X\,.\qquad(4.60)$$

Then T is an injective inclusion with dense image. Furthermore,

$$X^*:=\Big\{\big(u,v\big):\ u\in X_0^*,\ v\in X_0^*\Big\}\,,\qquad(4.61)$$

where

$$\left\langle\delta,h\right\rangle_{X\times X^*}=\left\langle\delta_0,h_0\right\rangle_{X_0\times X_0^*}+\left\langle\delta_1,h_1\right\rangle_{X_0\times X_0^*}\qquad\forall\delta=(\delta_0,\delta_1)\in X,\ \forall h=(h_0,h_1)\in X^*\,,\quad(4.62)$$

and

$$\|z\|_{X^*}:=\Big(\|u\|_{X_0^*}^2+\|v\|_{X_0^*}^2\Big)^{1/2}\qquad\forall z=(u,v)\in X^*\,.\qquad(4.63)$$

Moreover, the corresponding operator $\widetilde{T}\in\mathcal{L}(H;X^*)$, defined as in (2.12), is defined by

$$\widetilde{T}\cdot z=\big(\widetilde{\mathcal{T}}_0\cdot u,\widetilde{\mathcal{T}}_0\cdot v\big)=\big(S_0\cdot(L\cdot u),S_0\cdot(L\cdot v)\big)\qquad\forall z=(u,v)\in H\,.\qquad(4.64)$$

Thus $\{X,H,X^*\}$ is an evolution triple with the corresponding inclusion operators $T\in\mathcal{L}(X;H)$ and $\widetilde{T}\in\mathcal{L}(H;X^*)$, as it was defined in Definition 2.8.

Next let $\Lambda \in \mathcal{L}(H, X^*)$ be defined by

$$\Lambda \cdot z := \Big(-S_0 \cdot (\Delta v - v), S_0 \cdot (\Delta u - u) \Big) \qquad \forall z = (u, v) \in H$$
$$\text{(i.e. } (\Delta u - u) \in W^{-1,2}(\Omega, \mathbb{R}^k),\ (\Delta v - v) \in W^{-1,2}(\Omega, \mathbb{R}^k)) \,, \quad (4.65)$$

where S_0 is defined in (4.52). Then using (4.62) we deduce

$$\Big\langle h, \Lambda \cdot (T \cdot h) \Big\rangle_{X \times X^*} = -\Big\langle u, S_0 \cdot (\Delta v - v) \Big\rangle_{X_0 \times X_0^*} + \Big\langle v, S_0 \cdot (\Delta u - u) \Big\rangle_{X_0 \times X_0^*}$$
$$= -\int_\Omega \bigg\{ \Big((\nabla \Delta u)(x) - \nabla u(x) \Big) : \nabla v(x) + \Big(\Delta u(x) - u(x) \Big) \cdot v(x) \bigg\} dx$$
$$+ \int_\Omega \bigg\{ \Big((\nabla \Delta v)(x) - \nabla v(x) \Big) : \nabla u(x) + \Big(\Delta v(x) - v(x) \Big) \cdot u(x) \bigg\} dx = 0 \qquad \forall h = (u, v) \in X \,. \quad (4.66)$$

Furthermore, for $t \in [0, T_0]$ let $F_t(z) : H \to H$ be a function defined by

$$F_t(z) := \Big(\Theta\big(u(x,t), v(x,t), x, t\big) - v(x), \Xi\big(u(x,t), v(x,t), x, t\big) + u(x) \Big) \qquad \forall z = (u, v) \in H \,, \quad (4.67)$$

(we have $\Theta(u(x,t), v(x,t), x, t), \Xi(u(x,t), v(x,t), x, t) \in W_0^{1,2}(\Omega, \mathbb{R}^k)$ for a.e. t), i.e.

$$\Big\langle F_t(z), z_0 \Big\rangle_{H \times H} =$$
$$\int_\Omega \Bigg\{ \bigg(\nabla_x \Big\{ \Theta\big(u(x,t), v(x,t), x, t\big) \Big\} - \nabla v(x) \bigg) : \nabla u_0(x) + \Big(\Theta\big(u(x,t), v(x,t), x, t\big) - v(x) \Big) \cdot u_0(x)$$
$$+ \bigg(\nabla_x \Big\{ \Xi\big(u(x,t), v(x,t), x, t\big) \Big\} + \nabla u(x) \bigg) : \nabla v_0(x) + \Big(\Xi\big(u(x,t), v(x,t), x, t\big) + u(x) \Big) \cdot v_0(x) \Bigg\} dx$$
$$\forall z = (u, v) \in H,\ \forall z_0 = (u_0, v_0) \in H \,. \quad (4.68)$$

Then

$$\|F_t(z)\|_H \leq C \|z\|_H + f(t) \qquad \forall z \in H,\ \forall t \in [0, T_0] \,, \quad (4.69)$$

for some constant $C > 0$ and some $f(t) \in L^2(0, T_0; \mathbb{R})$. Furthermore, it satisfies the Lipschitz condition

$$\big\| \widetilde{T} \circ DF_t(z) \big\|_{\mathcal{L}(H; X^*)} \leq C \qquad \forall z \in H,\ \forall t \in [0, T_0] \,. \quad (4.70)$$

Moreover, since the embedding of $H = W_0^{1,2}(\Omega, \mathbb{R}^k)$ in $L^2_{loc}(\Omega, \mathbb{R}^k)$ is compact, we obtain that if $z_n \rightharpoonup z_0$ weakly in H then $z_n \to z_0$ strongly in $L^2_{loc}(\Omega, \mathbb{R}^k)$. Thus, by (4.69) we obtain $F_t(z_n) \rightharpoonup F_t(z_0)$ weakly in H. So F_t is weak to weak continuous in H. Then all the conditions of Corollary 3.1 satisfied and by this Corollary for every $w_0 \in W_0^{1,2}(\Omega, \mathbb{R}^k)$ and every $h_0 \in W_0^{1,2}(\Omega, \mathbb{R}^k)$ there exists $\zeta(t) = \big(u(x,t), v(x,t)\big) \in L^\infty(0, T_0; H)$, such that $\xi(t) := \widetilde{T} \cdot \big(\zeta(t)\big) \in W^{1,2}(0, T_0; X^*)$ and $\zeta(t)$ satisfy the following equation

$$\begin{cases} \frac{d\xi}{dt}(t) + \Lambda \cdot \zeta(t) + \widetilde{T} \cdot F_t\big(\zeta(t)\big) = 0 \quad \text{for a.e. } t \in (0, T_0) \,, \\ \zeta(0) = \big(w_0(x), h_0(x)\big) \,, \end{cases} \quad (4.71)$$

where we assume that $\zeta(t)$ is H-weakly continuous on $[0, T_0]$, as it was stated in Corollary 2.1. We can rewrite (4.71) as follows. Let $\big(u(x,t), v(x,t)\big) = \zeta(t)$. Then $u(x,t) \in L^\infty\big(0, T_0; W_0^{1,2}(\Omega, \mathbb{R}^k)\big)$, $v(x,t) \in L^\infty\big(0, T_0; W_0^{1,2}(\Omega, \mathbb{R}^k)\big)$, $u(x,t)$ and $v(x,t)$ are $W_0^{1,2}(\Omega, \mathbb{R}^k)$-weakly continuous on $[0, T_0]$, $u(x, 0) = w_0(x)$, $v(x, 0) = h_0(x)$ and by (4.55) and the definitions of Λ and F_t we obtain

$$-\bigg\langle \frac{\partial \delta}{\partial t}(x,t), S_0 \cdot u(x,t) \bigg\rangle_{X_0 \times X_0^*} + \bigg\langle \delta(x,t), S_0 \cdot \Big(-\Delta_x v(x,t) + \Theta\big(u(x,t), v(x,t), x, t\big) \Big) \bigg\rangle_{X_0 \times X_0^*} = 0$$
$$\forall \delta(x,t) \in C_c^1\big(0, T_0; X_0\big) \,, \quad (4.72)$$

$$-\left\langle\frac{\partial\delta}{\partial t}(x,t),S_0\cdot v(x,t)\right\rangle_{X_0\times X_0^*}+\left\langle\delta(x,t),S_0\cdot\Big(\Delta_x u(x,t)+\Xi\big(u(x,t),v(x,t),x,t\big)\Big)\right\rangle_{X_0\times X_0^*}$$
$$=0\qquad\forall\delta(x,t)\in C_c^1\big(0,T_0;X_0\big)\,. \quad (4.73)$$

Then, by Lemma 2.1 we obtain $\frac{du}{dt}(x,t)\in L^2\big(0,T_0;W^{-1,2}(\Omega,\mathbb{R}^k)\big)$ and $\frac{dv}{dt}(x,t)\in L^2\big(0,T_0;W^{-1,2}(\Omega,\mathbb{R}^k)\big)$ and thus $u(x,t)\in L^\infty\big(0,T_0;W_0^{1,2}(\Omega,\mathbb{R}^k)\big)\cap W^{1,2}\big(0,T_0;W^{-1,2}(\Omega,\mathbb{R}^k)\big)$ and $v(x,t)\in L^\infty\big(0,T_0;W_0^{1,2}(\Omega,\mathbb{R}^k)\big)\cap W^{1,2}\big(0,T_0;W^{-1,2}(\Omega,\mathbb{R}^k)\big)$. Moreover $\big(u(x,t),v(x,t)\big)$ solves (4.50). □

4.6 Incompressible Navier-Stokes equations and Magneto-Hydrodynamics

Let $\Omega\subset\mathbb{R}^N$ be a domain. The initial-boundary value problem for the incompressible Navier-Stokes Equations is the following one,

$$\begin{cases}(i)\ \ \frac{\partial v}{\partial t}+\operatorname{div}_x(v\otimes v)+\nabla_x p=\nu_h\Delta_x v+f\qquad\forall(x,t)\in\Omega\times(0,T_0)\,,\\(ii)\ \ \operatorname{div}_x v=0\qquad\forall(x,t)\in\Omega\times(0,T_0)\,,\\(iii)\ \ v(x,t)=\gamma(x,t)\qquad\forall(x,t)\in\partial\Omega\times(0,T_0)\,,\\(iv)\ \ v(x,0)=v_0(x)\qquad\forall x\in\Omega\,.\end{cases}\tag{4.74}$$

Here $v=v(x,t):\Omega\times(0,T_0)\to\mathbb{R}^N$ is an unknown velocity, $p=p(x,t):\Omega\times(0,T_0)\to\mathbb{R}$ is an unknown pressure, associated with v, $\nu_h>0$ is a given constant hydrodynamical viscosity, $f:\Omega\times(0,T_0)\to\mathbb{R}^N$ is a given force field $\gamma=\gamma(x,t)$ is a given velocity on the boundary (which can be nontrivial for fluid driven by its boundary) and $v_0:\Omega\to\mathbb{R}^N$ is a given initial velocity.

The initial-boundary value problem for the incompressible Magneto-Hydrodynamics is the following one,

$$\begin{cases}(i)\ \ \frac{\partial v}{\partial t}+\operatorname{div}_x(v\otimes v)-\operatorname{div}_x(b\otimes b)+\nabla_x p=\nu_h\Delta_x v+f\qquad\forall(x,t)\in\Omega\times(0,T_0)\,,\\(ii)\ \ \frac{\partial b}{\partial t}+\operatorname{div}_x(b\otimes v)-\operatorname{div}_x(v\otimes b)=\nu_m\Delta_x b\qquad\forall(x,t)\in\Omega\times(0,T_0)\,,\\(iii)\ \ \operatorname{div}_x v=0\qquad\forall(x,t)\in\Omega\times(0,T_0)\,,\\(iv)\ \ \operatorname{div}_x b=0\qquad\forall(x,t)\in\Omega\times(0,T_0)\,,\\(v)\ \ v(x,t)=0\qquad\forall(x,t)\in\partial\Omega\times(0,T_0)\,,\\(vi)\ \ b\cdot\boldsymbol{n}=0\qquad\forall(x,t)\in\partial\Omega\times(0,T_0)\,,\\(vii)\ \ \sum_{j=1}^N\Big(\frac{\partial b_i}{\partial x_j}-\frac{\partial b_j}{\partial x_i}\Big)\boldsymbol{n}_j=0\qquad\forall(x,t)\in\partial\Omega\times(0,T_0),\ \forall i=1,2,\ldots N\,,\\(viii)\ \ v(x,0)=v_0(x)\qquad\forall x\in\Omega\,,\\(ix)\ \ b(x,0)=b_0(x)\qquad\forall x\in\Omega\,.\end{cases}\tag{4.75}$$

Here $v=v(x,t):\Omega\times(0,T_0)\to\mathbb{R}^N$ is an unknown velocity, $b=b(x,t):\Omega\times(0,T_0)\to\mathbb{R}^N$ is an unknown magnetic field, $p=p(x,t):\Omega\times(0,T_0)\to\mathbb{R}$ is an unknown total pressure (hydrodynamical+magnetic), $\nu_h>0$ and $\nu_m>0$ are given constant hydrodynamical and magnetic viscosities, $f:\Omega\times(0,T_0)\to\mathbb{R}^N$ is a given force field, $v_0:\Omega\to\mathbb{R}^N$ is a given initial velocity, $b_0:\Omega\to\mathbb{R}^N$ is a given initial magnetic field and $\boldsymbol{n}$ is a normal to $\partial\Omega$.

Next if for some constant $\lambda\in\{0,1\}$ we consider the system:

$$\begin{cases}\frac{\partial v}{\partial t}+\operatorname{div}_x(v\otimes v)-\lambda\operatorname{div}_x(b\otimes b)+\nabla_x p=\nu_h\Delta_x v+f\qquad\forall(x,t)\in\Omega\times(0,T_0)\,,\\\frac{\partial b}{\partial t}+\lambda\operatorname{div}_x(b\otimes v)-\lambda\operatorname{div}_x(v\otimes b)=\nu_m\Delta_x b\qquad\forall(x,t)\in\Omega\times(0,T_0)\,,\\\operatorname{div}_x v=0\qquad\forall(x,t)\in\Omega\times(0,T_0)\,,\\\operatorname{div}_x b=0\qquad\forall(x,t)\in\Omega\times(0,T_0)\,,\\v(x,t)=\gamma(x,t)\qquad\forall(x,t)\in\partial\Omega\times(0,T_0)\,,\\b\cdot\boldsymbol{n}=0\qquad\forall(x,t)\in\partial\Omega\times(0,T_0)\,,\\\sum_{j=1}^N\Big(\frac{\partial b_i}{\partial x_j}-\frac{\partial b_j}{\partial x_i}\Big)\boldsymbol{n}_j=(\lambda/\nu_m)\big(\gamma\cdot\boldsymbol{n}\big)b\qquad\forall(x,t)\in\partial\Omega\times(0,T_0),\ \forall i=1,2,\ldots N\,,\\v(x,0)=v_0(x)\qquad\forall x\in\Omega\,,\\b(x,0)=b_0(x)\qquad\forall x\in\Omega\,,\end{cases}\tag{4.76}$$

then for $\lambda = 1$ and $\gamma \equiv 0$ this system will coincide with (4.75). On the other hand if (v, b, p) is a solution to (4.76) for $\lambda = 0$, then (v, p) will be a solution to (4.74).

If there exists a sufficiently regular function $r = r(x,t) : \Omega \times (0, T_0) \to \mathbb{R}^N$ such that $r(x,t) = \gamma(x,t)$ $\forall (x,t) \in \partial\Omega \times (0, T_0)$ and $\operatorname{div}_x r \equiv 0$, then fix it and define the new unknown function $u(x,t) := v(x,t) - r(x,t)$ and its initial value $u_0(x) := v_0(x) - r(x,0)$. Then we can rewrite (4.76) in the terms of (u, b, p) as

$$\begin{cases} \frac{\partial u}{\partial t} + \operatorname{div}_x \big(u \otimes u + r \otimes u + u \otimes r - \lambda b \otimes b\big) + \nabla_x p = \nu_h \Delta_x u + \hat{f} \quad \forall (x,t) \in \Omega \times (0, T_0)\,, \\ \frac{\partial b}{\partial t} + \lambda \operatorname{div}_x (b \otimes u - u \otimes b + b \otimes r - r \otimes b) = \nu_m \Delta_x b \qquad \forall (x,t) \in \Omega \times (0, T_0)\,, \\ \operatorname{div}_x u = 0 \qquad \forall (x,t) \in \Omega \times (0, T_0)\,, \\ \operatorname{div}_x b = 0 \qquad \forall (x,t) \in \Omega \times (0, T_0)\,, \\ u = 0 \qquad \forall (x,t) \in \partial\Omega \times (0, T_0)\,, \\ b \cdot \boldsymbol{n} = 0 \qquad \forall (x,t) \in \partial\Omega \times (0, T_0)\,, \\ \sum_{j=1}^N \left(\frac{\partial b_i}{\partial x_j} - \frac{\partial b_j}{\partial x_i} \right) \boldsymbol{n}_j = (\lambda/\nu_m)\big(r \cdot \boldsymbol{n}\big) b \qquad \forall (x,t) \in \partial\Omega \times (0, T_0),\ \forall i = 1, 2, \dots N\,, \\ u(x,0) = u_0(x) \qquad \forall x \in \Omega\,, \\ b(x,0) = b_0(x) \qquad \forall x \in \Omega\,, \end{cases} \tag{4.77}$$

where $\hat{f} := f + \Delta_x r - \partial_t r - div_x\,(r \otimes r)$. We will provide with the existence of solution for the system (4.77) for every constant $\lambda \in \{0, 1\}$.

We need some preliminaries.

Definition 4.1. Let $\Omega \subset \mathbb{R}^N$ be an open set. We denote:

- By $\mathcal{V}_N = \mathcal{V}_N(\Omega)$ the space $\{\varphi \in C_c^\infty(\Omega, \mathbb{R}^N) :\ div\,\varphi = 0\}$ and by $L_N = L_N(\Omega)$ the space, which is the closure of $\mathcal{V}_N$ in the space $L^2(\Omega, \mathbb{R}^N)$, endowed with the scalar product $\big\langle \varphi_1, \varphi_2 \big\rangle_{B_N} := \int_\Omega \varphi_1 \cdot \varphi_2 \, dx$ and the norm $\|\varphi\| := \big(\int_\Omega |\varphi|^2 dx \big)^{1/2}$.
- By $V_N = V_N(\Omega)$ the closure of $\mathcal{V}_N$ in $W_0^{1,2}(\Omega, \mathbb{R}^N)$ endowed with the scalar product $\big\langle \varphi_1, \varphi_2 \big\rangle_{V_N} := \int_\Omega \big(\nabla\varphi_1 : \nabla\varphi_2 + \varphi_1 \cdot \varphi_2\big) dx$ and the norm $\|\varphi\| := \big(\int_\Omega |\nabla\varphi|^2 dx + \int_\Omega |\varphi|^2 dx \big)^{1/2}$.
- $C_c^\infty(\overline{\Omega}, \mathbb{R}^N) := \big\{ \varphi : \Omega \to \mathbb{R}^N :\ \exists \bar{\varphi} \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N) \text{ s.t. } \bar{\varphi}(x) = \varphi(x)\ \forall x \in \Omega \big\}$.

Furthermore, given $\varphi \in \mathcal{D}'(\Omega, \mathbb{R}^N)$ denote

$$rot_x \varphi := \left\{ \frac{\partial \varphi_i}{\partial x_j} - \frac{\partial \varphi_j}{\partial x_i} \right\}_{1 \le i,j \le N} = \big(\nabla_x f\big) - \big(\nabla_x f\big)^T \in \mathcal{D}'\big(\Omega, \mathbb{R}^{N \times N}\big)\,, \tag{4.78}$$

and define the linear space

$$B'_N = B'_N(\Omega) := \left\{ \varphi \in L_N :\ rot_x \varphi \in L^2\big(\Omega, \mathbb{R}^{N \times N}\big) \right\}, \tag{4.79}$$

endowed with the scalar product $\big\langle \varphi_1, \varphi_2 \big\rangle_{B'_N} := \int_\Omega \big(\varphi_1 \cdot \varphi_2 + (1/2) rot_x \varphi_1 \cdot rot_x \varphi_2\big)\, dx$ and the corresponding norm $\|\varphi\|_{B'_N} := \big(< \varphi, \varphi >_{B'_N} \big)^{1/2}$. Then B'_N is a Hilbert space. Moreover, clearly B'_N is continuously embedded in $W_{loc}^{1,2}(\Omega, \mathbb{R}^N) \cap L_N$. We also denote a smaller space $B_N = B_N(\Omega)$ as a closure of $B'_N(\Omega) \cap C_c^\infty(\overline{\Omega}, \mathbb{R}^N)$ in $B'_N(\Omega)$ endowed with the norm of $B'_N(\Omega)$ (clearly if the boundary of domain Ω is sufficiently regular then B_N and B'_N coincide).

Proposition 4.5. *For every $r \in L^2\big(0, T_0; W^{1,2}(\Omega, \mathbb{R}^N)\big) \cap L^\infty$, $f \in L^2\big(0, T_0; L^2(\Omega, \mathbb{R}^N)\big)$, $g \in L^2\big(0, T_0; L^2(\Omega, \mathbb{R}^{N \times N})\big)$, $\nu_h > 0$, $\nu_m > 0$, $\lambda \in \{0, 1\}$, $v_0(\cdot) \in L_N$ and $b_0(\cdot) \in L_N$ there exist $u(x,t) \in L^2(0, T_0; V_N) \cap L^\infty(0, T_0; L_N)$ and $b(x,t) \in L^2(0, T_0; B_N) \cap L^\infty(0, T_0; L_N)$, such that $u(\cdot, t)$*

and $b(\cdot,t)$ *are* L_N*-weakly continuous in* t *on* $[0,T_0]$, $u(x,0)=v_0(x)$, $b(x,0)=b_0(x)$ *and* $u(x,t)$ *and* $b(x,t)$ *satisfy*

$$\int_0^{T_0}\int_\Omega\Bigg\{\Big(u(x,t)\otimes u(x,t)+r(x,t)\otimes u(x,t)+u(x,t)\otimes r(x,t)-\lambda b(x,t)\otimes b(x,t)+g(x,t)\Big):\nabla_x\psi(x,t)$$
$$-f(x,t)\cdot\psi(x,t)+u(x,t)\cdot\partial_t\psi(x,t)\Bigg\}\,dxdt=\int_0^{T_0}\int_\Omega\nu_h\nabla_x u(x,t):\nabla_x\psi(x,t)\,dxdt-\int_\Omega v_0(x)\cdot\psi(x,0)\,dx\,, \tag{4.80}$$

for every $\psi(x,t)\in C_c^1\big(\Omega\times[0,T_0),\mathbb{R}^N\big)\cap C^1\big([0,T_0];V_N\big)$ *and*

$$\int_0^{T_0}\int_\Omega\Bigg\{\lambda\Big(b(x,t)\otimes u(x,t)-u(x,t)\otimes b(x,t)+b(x,t)\otimes r(x,t)-r(x,t)\otimes b(x,t)\Big):\nabla_x\phi(x,t)$$
$$+b(x,t)\cdot\partial_t\phi(x,t)\Bigg\}\,dxdt=\int_0^{T_0}\int_\Omega\frac{\nu_m}{2}rot_x b(x,t):rot_x\phi(x,t)\,dxdt-\int_\Omega b_0(x)\cdot\phi(x,0)\,dx\,, \tag{4.81}$$

for every $\phi(x,t)\in C_c^1\big(\mathbb{R}^N\times[0,T_0),\mathbb{R}^N\big)\cap C^1\big([0,T_0];B_N\big)$*. I.e.*

$$\begin{cases}\frac{\partial u}{\partial t}+\operatorname{div}_x\big(u\otimes u+r\otimes u+u\otimes r-\lambda b\otimes b\big)+\nabla_x p=\nu_h\Delta_x u-f-\operatorname{div}_x g\quad\forall(x,t)\in\Omega\times(0,T_0)\,,\\ \frac{\partial b}{\partial t}+\lambda\operatorname{div}_x(b\otimes u-u\otimes b+b\otimes r-r\otimes b)=\nu_m\Delta_x b\qquad\forall(x,t)\in\Omega\times(0,T_0)\,,\\ \operatorname{div}_x u=0\qquad\forall(x,t)\in\Omega\times(0,T_0)\,,\\ \operatorname{div}_x b=0\qquad\forall(x,t)\in\Omega\times(0,T_0)\,,\\ u=0\qquad\forall(x,t)\in\partial\Omega\times(0,T_0)\,,\\ b\cdot\boldsymbol{n}=0\qquad\forall(x,t)\in\partial\Omega\times(0,T_0)\,,\\ rot_x b\cdot\boldsymbol{n}=(\lambda/\nu_m)\big(r\cdot\boldsymbol{n}\big)b\qquad\forall(x,t)\in\partial\Omega\times(0,T_0),\\ u(x,0)=u_0(x)\qquad\forall x\in\Omega\,,\\ b(x,0)=b_0(x)\qquad\forall x\in\Omega\,,\end{cases} \tag{4.82}$$

Moreover, if either $\lambda=0$ *and* Ω *is bounded or* $r(x,t)\equiv0$*, then* $u(x,t)$ *and* $b(x,t)$ *satisfy the energy inequality*

$$\frac12\int_\Omega\big|u(x,\tau)\big|^2dx+\frac12\int_\Omega\big|b(x,\tau)\big|^2dx+\int_0^\tau\int_\Omega\nu_h\big|\nabla_x u(x,t)\big|^2\,dxdt$$
$$+\int_0^\tau\int_\Omega\frac{\nu_m}{2}\big|rot_x b(x,t)\big|^2\,dxdt\le\frac12\int_\Omega\big|v_0(x)\big|^2dx+\frac12\int_\Omega\big|b_0(x)\big|^2dx$$
$$+\int_0^\tau\int_\Omega\Bigg(\Big\{g(x,t)+r(x,t)\otimes u(x,t)+u(x,t)\otimes r(x,t)\Big\}:\nabla_x u(x,t)$$
$$+\lambda\Big\{b(x,t)\otimes r(x,t)\Big\}:rot_x b(x,t)-f(x,t)\cdot u(x,t)\Bigg)\,dxdt\qquad\forall\tau\in[0,T_0]\,. \tag{4.83}$$

Proof. Fix $\nu_h>0$, $\nu_m>0$, $\lambda\in\{0,1\}$, $f\in L^2\big(0,T_0;L^2(\Omega,\mathbb{R}^N)\big)$ $g\in L^2\big(0,T_0;L^2(\Omega,\mathbb{R}^{N\times N})\big)$, $r\in L^2\big(0,T_0;W^{1,2}(\Omega,\mathbb{R}^N)\big)\cap L^\infty$, $v_0(\cdot)\in L_N$ and $b_0(\cdot)\in L_N$. Next define the space U'_N as a closure of $\mathcal{V}_N$ with respect to the norm

$$\|\varphi\|_{U'_N}:=\|\varphi\|_{V_N}+\sup_{x\in\Omega}|\varphi(x)|+\sup_{x\in\Omega}|\nabla\varphi(x)|\,. \tag{4.84}$$

and the space D'_N as a closure of $B_N\cap C_c^\infty(\overline{\Omega},\mathbb{R}^N)$ with respect to the norm

$$\|\varphi\|_{D'_N}:=\|\varphi\|_{B_N}+\sup_{x\in\Omega}|\varphi(x)|+\sup_{x\in\Omega}|\nabla\varphi(x)|\,. \tag{4.85}$$

Then clearly U'_N and D'_N are separable Banach spaces, which, however, are not reflexive. On the other hand, by Lemma 2.10 there exist separable Hilbert spaces U_N and D_N and bounded linear inclusion operators $A_1 \in \mathcal{L}(U_N; U'_N)$ and $A_2 \in \mathcal{L}(D_N; D'_N)$, such that A_1 and A_2 are injective, the image of A_1 is dense in U'_N and the image of A_2 is dense in B'_N On the other hand, clearly U'_N is trivially embedded in V_N, and the trivial embedding operator $I_1 \in \mathcal{L}(U'_N; V_N)$ is injective and has dense range in V_N. Similarly, D'_N is trivially embedded in B_N, and the trivial embedding operator $I_2 \in \mathcal{L}(D'_N; B_N)$ is injective and has dense range in B_N. Therefore if we define

$$Q_1 := I_1 \circ A_1 \in \mathcal{L}(U_N; V_N) \quad \text{and} \quad Q_2 := I_2 \circ A_2 \in \mathcal{L}(D_N; B_N)\,, \tag{4.86}$$

then Q_1 and Q_2 are injective and having dense ranges in V_N and B_N respectively. Next define $P_1 \in \mathcal{L}(V_N; L_N)$ as a trivial inclusion of V_N into L_N and $P_2 \in \mathcal{L}(B_N; L_N)$ as a trivial inclusion of B_N into L_N. Then clearly P_1 and P_2 are injective and having dense ranges in L_N. Finally define

$$\mathcal{T}_1 := P_1 \circ Q_1 \in \mathcal{L}(U_N; L_N) \quad \text{and} \quad \mathcal{T}_2 := P_2 \circ Q_2 \in \mathcal{L}(D_N; L_N)\,. \tag{4.87}$$

Then $\mathcal{T}_1$ and $\mathcal{T}_2$ are injective and having dense ranges in L_N. Next set

$$X := \Big\{ (\psi, \varphi) :\ \psi \in U_N,\ \varphi \in D_N \Big\}\,, \tag{4.88}$$

In this spaces we consider the norm

$$\|x\|_X := \sqrt{\|\psi\|^2_{U_N} + \|\varphi\|^2_{D_N}} \qquad \forall x = (\psi, \varphi) \in X\,. \tag{4.89}$$

Thus X is a separable reflexive Banach space. Similarly set

$$Z := \Big\{ \big(\psi(x), \varphi(x)\big) :\ \psi(x) : \Omega \to \mathbb{R}^N,\ \varphi(x) : \Omega \to \mathbb{R}^N,\ \psi(x) \in V_N,\ \varphi(x) \in B_N \Big\}\,, \tag{4.90}$$

In this spaces we consider the norm

$$\|z\|_Z := \sqrt{\|\psi\|^2_{V_N} + \|\varphi\|^2_{B_N}} \qquad \forall z = (\psi, \varphi) \in Z\,. \tag{4.91}$$

Thus Z is also a separable reflexive Banach space. Finally set

$$H := \Big\{ \big(\psi(x), \varphi(x)\big) :\ \psi(x) : \Omega \to \mathbb{R}^N,\ \varphi(x) : \Omega \to \mathbb{R}^N,\ \psi(x) \in L_N,\ \varphi(x) \in L_N \Big\}\,. \tag{4.92}$$

In this space we consider the scalar product

$$\begin{aligned} < h_1, h_2 >_{H \times H} &:= < \psi_1, \psi_2 >_{L_N \times L_N} + < \varphi_1, \varphi_2 >_{L_N \times L_N} \\ &= \int_\Omega \Big\{ \psi_1(x) \cdot \psi_2(x) + \varphi_1(x) \cdot \varphi_2(x) \Big\} dx \qquad \forall h_1 = (\psi_1, \varphi_1), h_2 = (\psi_2, \varphi_2) \in H\,. \end{aligned} \tag{4.93}$$

Then H is a Hilbert space. Furthermore, consider $Q \in \mathcal{L}(X, Z)$ by

$$Q \cdot h = \big(Q_1 \cdot \psi, Q_2 \cdot \varphi\big) \qquad \forall h = (\psi, \varphi) \in X\,. \tag{4.94}$$

Similarly set $P \in \mathcal{L}(Z, H)$ by

$$P \cdot z = \big(P_1 \cdot \psi, P_2 \cdot \varphi\big) \qquad \forall z = (\psi, \varphi) \in Z\,, \tag{4.95}$$

and consider $T \in \mathcal{L}(X, H)$ by

$$T \cdot h = \big(\mathcal{T}_1 \cdot \psi, \mathcal{T}_2 \cdot \varphi\big) \qquad \forall h = (\psi, \varphi) \in X\,, \tag{4.96}$$

Thus clearly $T = P \circ Q$ and T is an injective inclusion with dense image. Furthermore,

$$X^* := \Big\{ (\psi, \varphi) :\ \psi \in (U_N)^*,\ \varphi \in (D_N)^* \Big\}\,, \tag{4.97}$$

where

$$\big\langle\delta,h\big\rangle_{X\times X^*}=\big\langle\delta_0,h_0\big\rangle_{U_N\times(U_N)^*}+\big\langle\delta_1,h_1\big\rangle_{D_N\times(D_N)^*}\qquad\forall\delta=(\delta_0,\delta_1)\in X,\ \forall h=(h_0,h_1)\in X^*\,. \tag{4.98}$$

Thus $\{X,H,X^*\}$ is an evolution triple with the corresponding inclusion operators $T\in\mathcal{L}(X;H)$ and $\widetilde{T}\in\mathcal{L}(H;X^*)$, as it was defined in Definition 2.8. Next, let $\Phi(h):Z\to[0,+\infty)$ be defined by

$$\Phi(h):=\frac{1}{2}\int_\Omega\bigg(\nu_h\big|\nabla_x\psi(x)\big|^2+\frac{\nu_m}{2}\big|rot_x\varphi(x)\big|^2+\big|\psi(x)\big|^2+\big|\varphi(x)\big|^2\bigg)dx$$
$$\forall h=(\psi,\varphi)\in Z=\big(V_N,B_N\big)\,.$$

So the mapping $D\Phi(h):Z\to Z^*$ is linear and monotone. Furthermore, for every $t\in[0,T_0]$ let $\Theta_t(\sigma):H\to(U_N)^*$ be defined by

$$\Big\langle\delta,\Theta_t(\sigma)\Big\rangle_{U_N\times(U_N)^*}:=$$
$$-\int_\Omega\Bigg\{\bigg(w(x)\otimes w(x)+r(x,t)\otimes w(x,t)+w(x,t)\otimes r(x,t)-\lambda b(x)\otimes b(x)\bigg)+g(x,t)\Bigg\}:\nabla\{A_1\cdot\delta\}(x)\,dx$$
$$+\int_\Omega\Big(f(x,t)-w(x)\Big)\cdot\{A_1\cdot\delta\}(x)\,dx\qquad\forall\sigma=(w,b)\in L_N\oplus L_N\equiv H,\ \forall\delta\in U_N\,, \tag{4.99}$$

Next for every $t\in[0,T_0]$ let $\Xi_t(\sigma):H\to(D_N)^*$ be defined by

$$\Big\langle\delta,\Xi_t(\sigma)\Big\rangle_{D_N\times(D_N)^*}:=$$
$$-\int_\Omega\lambda\Big(b(x)\otimes w(x)-w(x)\otimes b(x)+b(x)\otimes r(x,t)-r(x,t)\otimes b(x)\Big):\nabla\{A_2\cdot\delta\}(x)\,dx$$
$$-\int_\Omega b(x)\cdot\{A_2\cdot\delta\}(x)\,dx\qquad\forall\sigma=(w,b)\in L_N\oplus L_N\equiv H,\ \forall\delta\in D_N\,, \tag{4.100}$$

Finally for every $t\in[0,T_0]$ let $F_t(\sigma):H\to X^*$ be defined by

$$F_t(\sigma):=\big(\Theta_t(\sigma),\Xi_t(\sigma)\big)\quad\forall\sigma\in H \tag{4.101}$$

Then $F_t(\sigma)$ is Gateaux differentiable at every $\sigma\in H$, and the derivative of $F_t(\sigma)$ satisfy the condition

$$\|DF_t(\sigma)\|_{\mathcal{L}(H;X^*)}\le C\big(\|\sigma\|_H+1\big)\quad\forall\sigma\in H,\ \forall t\in[0,T_0]\,, \tag{4.102}$$

for some constant $C>0$. Moreover,

$$\Big\langle\delta,F_t(T\cdot\delta)\Big\rangle_{X\times X^*}=\Big\langle\psi,\Theta_t(T\cdot\delta)\Big\rangle_{U_N\times(U_N)^*}+\Big\langle\varphi,\Xi_t(T\cdot\delta)\Big\rangle_{D_N\times(D_N)^*}=$$
$$-\int_\Omega\Bigg\{\bigg(w(x)\otimes w(x)+r(x,t)\otimes w(x,t)+w(x,t)\otimes r(x,t)-\lambda b(x)\otimes b(x)\bigg)+g(x,t)\Bigg\}:\nabla w(x)\,dx$$
$$+\int_\Omega\Big(f(x,t)-w(x)\Big)\cdot w(x)\,dx-\int_\Omega\lambda\Big(b(x)\otimes w(x)-w(x)\otimes b(x)+b(x)\otimes r(x,t)-r(x,t)\otimes b(x)\Big):\nabla b(x)\,dx$$
$$-\int_\Omega b(x)\cdot b(x)\,dx\qquad\text{where }w=A_1\cdot\psi,\ b=A_2\cdot\varphi\quad\forall\delta=(\psi,\varphi)\in U_N\oplus D_N=X,\ \forall t\in[0,T_0]\,, \tag{4.103}$$

Thus since $w = A_1 \cdot \psi \in U'_N$ and $b = A_2 \cdot \varphi \in D'_N$ we rewrite (4.103) as follows,

$$\Big\langle \delta, F_t(T\cdot\delta)\Big\rangle_{X\times X^*} = \int_\Omega \Big(f(x,t)\cdot w(x) - g(x,t) : \nabla w(x)\Big)\,dx - \int_\Omega \Big(\big|w(x)\big|^2 + \big|b(x)\big|^2\Big)\,dx$$
$$- \int_\Omega \bigg(\Big\{r(x,t)\otimes w(x) + w(x)\otimes r(x,t)\Big\} : \nabla w(x) + \lambda\Big\{b(x)\otimes r(x,t)\Big\} : rot_x b(x)\bigg)dx$$
$$- \int_\Omega \frac{1}{2}\bigg\{w(x)\cdot\nabla_x\big|w(x)\big|^2 + \lambda w(x)\cdot\nabla_x\big|b(x)\big|^2 - 2\lambda b(x)\cdot\nabla_x\Big(w(x)\cdot b(x)\Big)\bigg\}\,dx$$
$$\text{where } w = A_1\cdot\psi,\ b = A_2\cdot\varphi \quad \forall\delta = (\psi,\varphi)\in U_N\oplus D_N = X,\ \forall t\in[0,T_0]\,, \tag{4.104}$$

On the other hand $w(x), b(x) \in L_N$ and thus $\mathrm{div}_x\{\chi_\Omega w\} = \mathrm{div}_x\{\chi_\Omega b\}$ in the sense of distributions (here χ_Ω is characteristic function of the set Ω). Thus the last integral in (4.104) vanishes, and therefore, since $r(x,t)\in L^\infty$ we obtain

$$\Big\langle \delta, F_t(T\cdot\delta)\Big\rangle_{X\times X^*} = \int_\Omega \Big(f(x,t)\cdot w(x) - g(x,t) : \nabla w(x)\Big)\,dx - \int_\Omega \Big(\big|w(x)\big|^2 + \big|b(x)\big|^2\Big)\,dx$$
$$- \int_\Omega \bigg(\Big\{r(x,t)\otimes w(x) + w(x)\otimes r(x,t)\Big\} : \nabla w(x) + \lambda\Big\{b(x)\otimes r(x,t)\Big\} : rot_x b(x)\bigg)dx \geq$$
$$-C\Big(\big\|Q\cdot\delta\big\|_Z + 1\Big)\Big(\big\|T\cdot\delta\big\|_H + 1\Big) - \mu(t) \quad \text{where } w = A_1\cdot\psi,\ b = A_2\cdot\varphi \quad \forall\delta \in= (\psi,\varphi)\in X,\ \forall t\in[0,T_0]. \tag{4.105}$$

Here $\mu(t)\in L^1(0,T_0;\mathbb{R})$ is some nonnegative function.

Next consider the sequence of open sets $\{\Omega_j\}_{j=1}^\infty$ such that for every $j\in\mathbb{N}$, Ω_j is compactly embedded in Ω_{j+1}, and $\cup_{j=1}^\infty\Omega_j = \Omega$. Then set $Z_j := L^2\big(\Omega_j,\mathbb{R}^N\big)$ and $\bar{L}_j \in \mathcal{L}(L_N, Z_j)$ by

$$\bar{L}_j\cdot\big(h(x)\big) := h(x)\llcorner\Omega_j \in L^2\big(\Omega_j,\mathbb{R}^N\big) = Z_j \qquad \forall h(x)\in L_N(\Omega)\,.$$

Thus, by the standard embedding theorems in the Sobolev Spaces, the operators $\bar{L}_j\circ P_1 \in \mathcal{L}(V_N, Z_j)$ and $\bar{L}_j\circ P_2\in\mathcal{L}(B_N,Z_j)$ are compact for every j. Moreover, if $\{\sigma_n\}_{n=1}^\infty\subset H$ is a sequence such that $\sigma_n = (h_n,w_n)\rightharpoonup\sigma_0 = (h_0,w_0)$ weakly in H and $\bar{L}_j\cdot h_n\to\bar{L}_j\cdot h_0$ and $\bar{L}_j\cdot w_n\to\bar{L}_j\cdot w_0$, strongly in Z_j as $n\to+\infty$ for every j, then we have $h_n\to h_0$ and $w_n\to w_0$ strongly in $L^2_{loc}(\Omega,\mathbb{R}^N)$ and thus, by (4.101) and (4.102) we must have $F_t(\sigma_n)\rightharpoonup F_t(\sigma_0)$ weakly in X^*.

Thus all the conditions of Theorem 3.5 are satisfied. Applying this Theorem we deduce that there exists a function $h(t)\in L^2\big(0,T_0;Z\big)$ such that $\sigma(t) := P\cdot h(t)$ belongs to $L^\infty(0,T_0;H)$, $\gamma(t) := \widetilde{T}\cdot\sigma(t)$ belongs to $W^{1,2}(0,T_0;X^*)$ and $h(t)$ is a solution to

$$\begin{cases} \frac{d\gamma}{dt}(t) + F_t\big(\sigma(t)\big) + Q^*\cdot D\Phi\big(h(t)\big) = 0 \quad \text{for a.e. } t\in(0,T_0)\,,\\ \sigma(0) = \big(v_0(x), b_0(x)\big)\,, \end{cases} \tag{4.106}$$

where we assume that $\sigma(t)$ is H-weakly continuous on $[0,T_0]$ and $Q^*\in\mathcal{L}(Z^*,X^*)$ is the adjoint to Q operator. Then by the definitions of Φ and F_t, $h(x,t) := \big(u(x,t), b(x,t)\big)$ satisfies that $u(x,t)\in L^2(0,T_0;V_N)\cap L^\infty(0,T_0;L_N)$ and $b(x,t)\in L^2(0,T_0;B_N)\cap L^\infty(0,T_0;L_N)$, $u(\cdot,t)$ and $b(\cdot,t)$ are L_N-weakly continuous in t on $[0,T_0]$, $u(x,0) = v_0(x)$, $b(x,0) = b_0(x)$ and $u(x,t)$ and $b(x,t)$ satisfy

$$\int_0^{T_0}\int_\Omega\Bigg\{\Big(u(x,t)\otimes u(x,t) + r(x,t)\otimes u(x,t) + u(x,t)\otimes r(x,t) - \lambda b(x,t)\otimes b(x,t) + g(x,t)\Big) : \nabla_x\big\{A_1\cdot\psi(t)\big\}(x)$$
$$- f(x,t)\cdot\big\{A_1\cdot\psi(t)\big\}(x) + u(x,t)\cdot\big\{A_1\cdot\partial_t\psi(t)\big\}(x)\Bigg\}\,dxdt$$
$$= \int_0^{T_0}\int_\Omega \nu_h\nabla_x u(x,t) : \nabla_x\big\{A_1\cdot\psi(t)\big\}(x)\,dxdt - \int_\Omega v_0(x)\cdot\big\{A_1\cdot\psi(0)\big\}(x)\,dx\,, \tag{4.107}$$

for every $\psi(t) \in C^1\big([0,T_0];U_N\big)$ such that $\psi(T_0)=0$ and

$$\int_0^{T_0}\int_\Omega\Bigg\{\lambda\Big(b(x,t)\otimes u(x,t)-u(x,t)\otimes b(x,t)+b(x,t)\otimes r(x,t)-r(x,t)\otimes b(x,t)\Big) : \nabla_x\big\{A_2\cdot\phi(t)\big\}(x)$$
$$+\, b(x,t)\cdot\big\{A_2\cdot\partial_t\phi(t)\big\}(x)\Bigg\}\,dxdt$$
$$=\int_0^{T_0}\int_\Omega \frac{\nu_m}{2} rot_x b(x,t) : rot_x\big\{A_2\cdot\phi(t)\big\}(x)\,dxdt-\int_\Omega b_0(x)\cdot\big\{A_2\cdot\phi(0)\big\}(x)\,dx\,, \quad (4.108)$$

for every $\phi(t) \in C^1\big([0,T_0];D_N\big)$ such that $\phi(T_0)=0$. Thus since the image of A_1 is dense in U'_N and the image of A_2 is dense in D'_N, we deduce that $u(x,t)$ and $b(x,t)$ are solutions of (4.80) and (4.81).

Next by (4.105) and by the definition of Φ we have

$$\Big\langle\delta, Q^*\cdot D\Phi(Q\cdot\delta)+F_t(T\cdot\delta)\Big\rangle_{X\times X^*}=$$
$$\int_\Omega\bigg(\nu_h\big|\nabla_x w(x)\big|^2+\frac{\nu_m}{2}\big|rot_x b(x)\big|^2+\int_\Omega\Big(f(x,t)\cdot w(x)-g(x,t):\nabla w(x)\Big)\,dx$$
$$-\int_\Omega\bigg(\big\{r(x,t)\otimes w(x)+w(x)\otimes r(x,t)\big\}:\nabla w(x)+\lambda\big\{b(x)\otimes r(x,t)\big\}:rot_x b(x)\bigg)dx$$
$$\text{where } w=A_1\cdot\psi,\ b=A_2\cdot\varphi\quad \forall\delta=(\psi,\varphi)\in X,\ \forall t\in[0,T_0]\,. \quad (4.109)$$

However, if Ω is bounded then the embedding operator P_1 is compact. On the other hand, either $\lambda=0$ and Ω is bounded or $r(x,t)\equiv 0$. Thus, by (4.109) together with Theorem 3.5, we finally deduce (4.83). □

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